PHYSNET

MISN-0-513

MAXWELL'S EQUATIONS

Electricity and Magnetism

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

MAXWELL'S EQUATIONS

by

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Title: **Maxwell's Equations**

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Input Skills:

1. Vocabulary: frequency, wavelength, phase (MISN-0-201); relaxation time, constitutive.
2. Determine the electrostatic energy density in a region of space given the electric fields present (MISN-0-508).
3. Determine the magnetostatic energy density in a region of space given the magnetic fields present (MISN-0-512).
4. Use complex notation to represent harmonic functions of position and time.

Output Skills (Knowledge):

K1. State Maxwell's equations in differential form and the three constitutive equations.

K2. State the equation of conservation of electromagnetic energy, explaining the meaning of each term.

K3. Derive the homogeneous wave equation for an electromagnetic wave propagating in a linear, homogeneous, isotropic medium.

K4. Derive the boundary conditions on the field vectors at the interface of two media of given conductivities.

Output Skills (Problem Solving):

S1. Given a non-static distribution of free charge, find the displacement current and use Ampere's law to determine the resulting magnetic field.

S2. Given the electric field, the magnetic field, or the Poynting vector, use Maxwell's equations to determine the other two vector quantities.

External Resources (Required):

1. J. Reitz, F. Milford and R. Christy, *Foundations of Electromagnetic Theory*, 4th Edition, Addison-Wesley (1993).

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New authors, reviewers and field testers are welcome.

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1. Introduction

Before beginning this Unit, the following is a collection of equations from previous Units.

(a) Gauss' Law;

$$\vec{\nabla} \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t) \tag{1}$$

where

$$\vec{D}(\vec{r},t) = \epsilon \vec{E}(\vec{r},t) \tag{2}$$

(b) Ampere's Law:

$$\vec{\nabla} \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) \tag{3}$$

where

$$\vec{B}(\vec{r},t) = \mu \vec{H}(\vec{r},t) \tag{4}$$

(c) Faraday's Law:

$$\vec{\nabla} \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t} \tag{5}$$

(d) Divergenceless Magnetic Induction Vector:

$$\vec{\nabla} \cdot \vec{B}(\vec{r},t) = 0 \tag{6}$$

(e) Conservation of Charge or Equation of Continuity:

$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \vec{\nabla} \cdot \vec{J}(\vec{r},t) = 0 \tag{7}$$

(f) Force on a charge q moving with velocity $\vec{v}$ in an electric and magnetic field:

$$\vec{F}(\vec{r},t) = q(\vec{E}(\vec{r},t) + \vec{v} \times \vec{B}(\vec{r},t)) \tag{8}$$

Eq. (7) is sometimes called the Lorentz force equation. Also, an explicit time dependence is allowed in each of the physical quantities.

As indicated in the last Unit, some of these equations are inconsistent. That is, eqs. (1), (3), and (7) imply that

$$\frac{\partial \rho}{\partial t} = 0, \text{ and } \frac{\partial \vec{D}}{\partial t} = 0 \tag{9}$$

These results are obviously too restrictive since time-varying charge densities and displacement vectors are real possibilities. One way out of this situation is if one of the equations is incomplete. A candidate would be Ampere's Law. Let us assume that Ampere's Law reads

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{\alpha} \tag{10}$$

Then,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{\alpha}$$

so

$$\vec{\nabla} \cdot \vec{J} = -\vec{\nabla} \cdot \vec{\alpha} \tag{11}$$

But, the equation of continuity implies that

$$\frac{\partial \rho}{\partial t} = \vec{\nabla} \cdot \vec{\alpha} \tag{12}$$

and Gauss' Law finally yields

$$\vec{\nabla} \cdot \left(\frac{\partial \vec{D}}{\partial t} \right) = \vec{\nabla} \cdot \vec{\alpha} \tag{13}$$

Thus, if we choose

$$\frac{\partial \vec{D}}{\partial t} = \vec{\alpha} \tag{14}$$

consistency among the three equations in question without the restrictions of eq. (9) can be obtained. Thus, Ampere's Law in complete form is

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

The term $\partial \vec{D}/\partial t$ has the dimension of a current density so that it is sometimes called a displacement current density. The displacement current has real experimental implications which have been shown to be valid.

The following set of equations is usually referred to as Maxwell's equations.

$$\vec{\nabla} \cdot \vec{D} = \rho, \ \ \vec{\nabla} \cdot \vec{B} = 0 \tag{15}$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Sometimes the term Maxwell's equations also includes the following set,

$$\vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H} \tag{16}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0, \quad \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

These eight equations give a complete description of dynamic as well as static electromagnetic phenomena. I will refer to these eight equations (15) and (16) as Maxwell's equations.

2. Procedures

1. Read Ch. 16, Sec. 16 - 1 to 16 - 6.
2. Write down and memorize the eight Maxwell's equations given in eqs. (15) and (16) of the Introduction. Be able to identify the equations by the names indicated in Output Skill K2.
3. Write down and memorize the derivations called for in Output Skill K3.
4. Write down and memorize the explanations of the terms in eq. (16-20) The explanations are given in the text.
5. Solve these problems:

Problem	**Type**
16 - 1	Displacement current
16 - 10, 16 - 12	Solutions to Maxwell's equations under various conditions
16 - 4	Poynting vector

Note: When the electromagnetic fields are complex, the energy density and Poynting vector $\vec{S}$ should be replaced by,

$$\omega = \frac{1}{2}(\vec{D}^* \cdot \vec{E} + \vec{B} \cdot \vec{H}^*)$$

and,

$$\vec{S} = \vec{E} \times \vec{H}^*$$

The observable electromagnetic fields, energy density, and Poynting vector are then obtained by taking the real part (or imaginary part) of the appropriate quantity. Thus,

$$(\vec{S})_{\text{observed}} = Re(\vec{S}) = Re(\vec{E} \times \vec{H}^*)$$

The asterisk denotes complex configuration.

Acknowledgments

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