

MAGNETIC MATERIALS AND  
BOUNDARY VALUE PROBLEMS

# Electricity and Magnetism

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by  
C. P. Frahm

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Title: **Magnetic Materials and Boundary Value Problems**

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Version: 10/18/2001

Evaluation: Stage B0

Length: 2 hr; 12 pages

**Input Skills:**

1. Vocabulary: magnetization, magnetic intensity, surface current density, transport current, magnetic scalar potential, ferromagnetic material, pole densities (MISN-0-510).
2. Solve electric circuit problems including series and parallel resistance combinations (MISN-0-119).
3. Use Ampere's law with systems of sufficient symmetry to determine the magnetic intensity (MISN-0-510).

**Output Skills (Knowledge):**

- K1. Vocabulary: magnetic flux, the boundary conditions on the magnetic induction and the magnetic intensity, magnetic circuit, magnetomotive force, reluctance, demagnetizing factor, magnetic shielding factor.
- K2. Derive Laplace's equation for the magnetic scalar potential for linear magnetic material starting from the basic field equations for magnetostatics when current density is zero.

**Output Skills (Problem Solving):**

- S1. Use Ampere's law, boundary value conditions and the hysteresis curves of any permanent magnet present to determine the magnetic induction in a given magnetic circuit.
- S2. Given a magnetic material of spherical or cylindrical symmetry, either in an external field or with a permanent magnetization and with no transport current present, determine the magnetic fields inside and outside the material.

**External Resources (Required):**

1. J. Reitz, F. Milford and R. Christy, *Foundations of Electromagnetic Theory*, 4th Edition, Addison-Wesley (1993).

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## 1. Introduction

The purpose of this Unit is to solve problems involving materials with different magnetic characteristics. To do this, the behavior of the field vectors  $\vec{B}$  and  $\vec{H}$  in passing an interface between two media must be known. Boundary conditions are derived for  $\vec{B}$  and  $\vec{H}$ , analogous to the boundary conditions on the field vectors  $\vec{E}$  and  $\vec{D}$  in electrostatics. It will be seen that the normal component of  $\vec{B}$  across an interface must be continuous while the tangential component of  $\vec{H}$  across the interface is discontinuous. The discontinuity in  $\vec{H}$  is proportional to the true surface current density.

The problems analyzed in this Unit include:

1. Toroidal current winding units a ferromagnetic core with and without an air gap
2. Other magnetic circuits with a well-confined flux and simple geometry
3. Magnetic fields for magnetic materials with no transport currents and with the flux not well-confined. Examples include spheres and cylinders in an external field with no permanent magnetization  $\vec{M}$  as well as permanently magnetized spheres and cylinders with no external field. These examples, will allow us to introduce the concepts of magnetic shielding and demagnetization factor, respectively.

In order to deal with magnetic circuits, the magnetomotive force and reluctance are introduced. In the more general case of flux which is not well-confined, the magnetic scalar potential which satisfies Laplace's equation with no transport currents present is utilized.

## 2. Procedures

1. Review the concept of magnetic flux  $\Phi$  in Sec. 8-9 of the text.
2. Read Secs. 9-7 to 9-11 of the text. Read the *Supplementary Notes*.

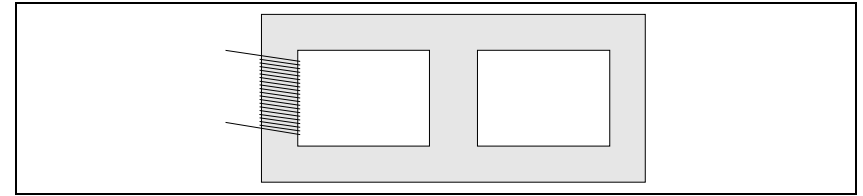


Figure 1. .

3. Write down or underline in the text the definitions and concepts asked for in Output Skill K1.
4. Solve these problems (note that for Problem 9-18, Figure 9-21 in the text should be replaced by our Fig. 1):

Problem	Type
9-9, 9-10	Magnetic induction for toroidal current windings with a material core
9-18,9-19	Simple magnetic circuits
9-16	Magnetic shielding factor of a cylindrical shell

## 3. Supplementary Notes

### 1. Magnetic Shielding: Spherical Shell of Permeable Material in a Uniform Field

In Fig. 2, since there are no conduction currents present,

$$\vec{\nabla} \times \vec{H} = 0 \Rightarrow \vec{H} = -\vec{\nabla}V^*$$

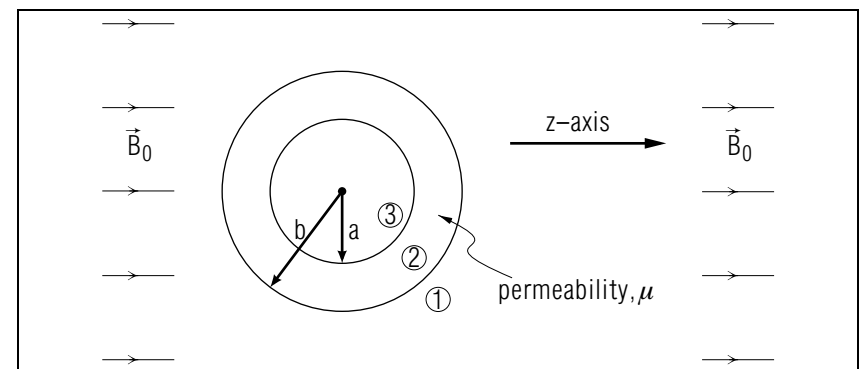


Figure 2. .

Since,  $\vec{B} = \mu\vec{H}$ ,  $\vec{\nabla} \cdot \vec{B} = 0$  becomes  $\vec{\nabla} \cdot \vec{H} = 0$  in the various regions. Hence,

$$\nabla^2 V^* = 0$$

everywhere. Thus for  $r > b$ ,

$$V_1^* = -B_0 r \frac{\cos \theta}{\mu_0} + \sum_{\ell=0}^{\infty} \frac{\alpha_{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta)$$

in order that  $\vec{H} \rightarrow \vec{B}_0/\mu_0$  as  $r \rightarrow \infty$ . For the inner regions,

$$(a < r < b) \quad V_2^* = \sum_{\ell=0}^{\infty} \left( \beta_{\ell} r^{\ell} + \gamma_{\ell} \frac{1}{r^{\ell+1}} \right) P_{\ell}(\cos \theta)$$

$$(r < a) \quad V_3^* = \sum_{\ell=0}^{\infty} \delta_{\ell} r^{\ell} P_{\ell}(\cos \theta)$$

since  $V^*$  must be finite at  $r = 0$ . The boundary conditions at  $r = a$  and  $r = b$  are that  $H_{\theta}$  and  $B_r$  be continuous. So,

$$\begin{aligned} \left. \frac{\partial V_1^*}{\partial \theta} \right|_{r=b} &= \left. \frac{\partial V_2^*}{\partial \theta} \right|_{r=b}, \quad \left. \frac{\partial V_2^*}{\partial \theta} \right|_{r=a} = \left. \frac{\partial V_3^*}{\partial \theta} \right|_{r=a} \\ \mu_0 \left. \frac{\partial V_1^*}{\partial r} \right|_{r=b} &= \mu \left. \frac{\partial V_2^*}{\partial r} \right|_{r=b}, \quad \mu \left. \frac{\partial V_2^*}{\partial r} \right|_{r=a} = \mu_0 \left. \frac{\partial V_3^*}{\partial r} \right|_{r=a} \end{aligned}$$

These four conditions which hold for all angles  $\theta$  are sufficient to determine the unknown constants. All coefficients with  $\ell = 1$  vanish. The  $\ell = 1$  coefficients satisfy the four simultaneous equations,

$$\begin{aligned} \alpha_1 - b^3 \beta_1 - \gamma_1 &= b^3 \frac{B_0}{\mu_0} \\ 2\alpha_1 \frac{\mu}{\mu_0} b^3 \beta_1 - 2 \frac{\mu}{\mu_0} \gamma_1 &= -b^3 \frac{B_0}{\mu_0} \\ a^3 \beta_1 + \gamma_1 - a^3 \delta_1 &= 0 \\ \frac{\mu}{\mu_0} a^3 \beta_1 - 2 \frac{\mu}{\mu_0} \gamma_1 - a^3 \delta_1 &= 0 \end{aligned}$$

The solutions for  $\alpha_1$  and  $\delta_1$  are,

$$\alpha_1 = \left[ \frac{(2\mu + \mu_0)(\mu - \mu_0)}{(2\mu + \mu_0)(\mu + 2\mu_0) - 2a^3/b^3(\mu - \mu_0)^2} \right] (b^3 - a^3)(B_0/\mu_0)$$

and

$$\delta_1 = - \left[ \frac{9\mu\mu_0}{(2\mu + \mu_0)(\mu + 2\mu_0) - 2a^3/b^3(\mu - \mu_0)^2} \right] (B_0/\mu_0)$$

The potential outside the spherical shell corresponds to a uniform field  $\vec{B}_0$  plus a dipole field with dipole moment  $\alpha_1$  oriented parallel to  $\vec{B}_0$ . Inside the cavity, there is a uniform magnetic force field parallel to  $\vec{B}_0$  and equal in magnitude to  $-\delta_1$ . For  $\mu \gg \mu_0$  the dipole moment  $\alpha_1$  and the inner field  $-\delta_1$  become,

$$\begin{aligned} \alpha_1 &\rightarrow b^3(B_0/\mu_0) \\ -\delta_1 &\rightarrow \frac{9\mu_0}{2\mu \left(1 - \frac{a^3}{b^3}\right)} (B_0/\mu_0) \end{aligned}$$

Thus, the inner field is proportional to  $(\mu/\mu_0)^{-1}$ . Consequently, a shield made of high-permeability material with  $\mu/\mu_0 \sim 10^3$  to  $10^6$  causes a great reduction in the field inside, even for a relatively thin shell. Thus, the magnetic induction in the cavity is given by,

$$B_3 = \mu_0 H_3 = -\mu_0 - \delta_1$$

The magnetic shielding factor,

$$h_m \equiv \frac{B_0}{B_3} = -\frac{B_0}{\mu_0 \delta_1} \approx \frac{2}{9} k_m \left(1 - \frac{a^3}{b^3}\right)$$

if  $k_m \gg 1$  (see Fig. 3.).

## 2. Correct Derivation of eqs. (9-68) and (9-69):

Equation (9-67) of the text is not correct as the text states. The text then pulls (9-68) out of the air. These are indeed the correct equations, but they can be derived from basic principles rather than being plucked from thin air. The principles are two:

- (1) Ampere's circuital law

Thus,

$$\oint \vec{H}_t = H_{1t} \cdot \ell + H_{2t}(\text{gap}) \cdot d + H_{2t}(\text{material}) \cdot (\ell - d) = NI$$

in an obvious notation. But,

$$H_{1t} = \frac{NI}{\ell}$$

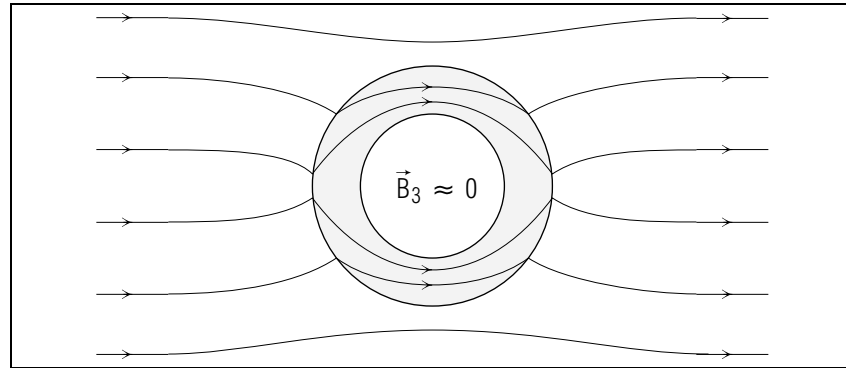


Figure 3. .

so that,

$$H_{2t}(\text{gap}) \cdot \frac{d}{\ell} + H_{2t}(\text{material}) \cdot \left(1 - \frac{d}{\ell}\right) = 0 \quad (1)$$

- (2) The boundary condition on the magnetic induction at the interface between the material and the air in the gap. Thus,

$$B_t(\text{gap}) = B_t(\text{material}) \quad (2)$$

But,

$$B_t(\text{gap}) = \mu_0 H_t(\text{gap}) \quad (3)$$

and

$$B_t(\text{material}) = \mu_t (H_t(\text{material}) + M_t) \quad (4)$$

Combining eqs. (2), (3), and (4) gives

$$H_t(\text{gap}) = H_t(\text{material}) + M_t \quad (5)$$

Combining eqs. (1) and (5) gives

$$H_{2t}(\text{material}) = -M_t \frac{d}{\ell}$$

and

$$H_{2t}(\text{gap}) = M_t \left(1 - \frac{d}{\ell}\right)$$

## Acknowledgments

The author would like to thank Illinois State University for support in the construction of this lesson. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

