PHYSNET

MISN-0-505

SOLUTIONS TO LAPLACE'S EQUATION

Electricity and Magnetism

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

SOLUTIONS TO LAPLACE'S EQUATION

by

R. D. Young

Title: **Solutions to Laplace's Equation**

Author: R. D. Young, Dept. of Physics, Ill. State Univ.

Version: 2/1/2000 Evaluation: Stage B0

Length: 2 hr; 13 pages

Input Skills:

1. Vocabulary: volume charge density, surface charge density (MISN-0-504); particular solution, boundary conditions (MISN-0-486).
2. State Gauss's law in differential form (MISN-0-504).
3. State the relation between electric field and potential (MISN-0-504).
4. Solve simple ordinary differential equations.

Output Skills (Knowledge):

K1. Derive Poisson's and Laplace's equations from the differential form of Gauss's law. Define each symbol appearing in these two equations.

K2. State the Laplacian operator in rectangular, spherical and cylindrical coordinates.

K3. State the superposition theorem and the uniqueness theorem relevant to solutions of Laplace's equation. Explain the usefulness of these two theorems.

Output Skills (Problem Solving):

S1. Given a particular arrangement of symmetric equipotential surfaces, use Laplace's equation to determine the potential in a given region of space.

S2. Determine the potential outside a symmetric conductor situated in a uniform electric field using zonal or cylindrical harmonics. Use the determined potential to calculate the surface change density on the conductor.

External Resources (Required):

1. J. Reitz, F. Milford and R. Christy, *Foundations of Electromagnetic Theory*, 4th Edition, Addison-Wesley (1993).

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

SOLUTIONS TO LAPLACE'S EQUATION

by
R. D. Young

2. Introduction

The solution to an electrostatic problem is complete when the charge distribution is everywhere specified. For then, the potential and electric fields are given directly as integrals over this charge distribution:

$$U(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq'}{|\vec{r} - \vec{r'}|}$$

or

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3}\, dq'$$

For complicated charge distributions, these integrals can be very difficult, if not impossible, to do analytically. Approximate numerical calculations can be carried out by hand or by digital computer. However, in cases where some of the charge distribution is not known, this is even impossible. One example would be any problem where conductors are involved. Although the potential or total charge on the conductors may be known, the exact distribution of charge may not be known.

In order to handle such problems, alternative methods for calculating the potential and / or electrical field are required. This unit develops one of these methods. The method is essentially a means to obtain the potential $U(\vec{r})$ in charge-free space. The electrostatic field can then be calculated as well as the charge distribution on all conductors.

The method uses the fundamental *differential equation* (as opposed to integral equation) for the potential $U(\vec{r})$. In charge-free space, this equation reads:

$$\nabla^2 U(\vec{r}) = \nabla^2 U(x, y, z) = 0\,.$$

The operator ∇^2 is called the "Laplacian." The problems in this section will have certain symmetries which can he exploited in order to simplify the computations. The two cases treated in this unit involve the symmetry of the geometries with respect to reflection through the origin and through the z-axis. Certain types of coordinate systems are especially appropriate when these symmetries are present. The spherical coordinate

system is generally used when the geometry is symmetric with respect to reflection through the origin (sometimes the phrase "symmetric about the origin" is used). The cylindrical coordinate system is generally used when the geometry is symmetric with respect to reflection through the z-axis ("symmetric about the z-axis"). Thus, the "Laplacian" will need to be expressed in each of these coordinate systems. The method of solution results in the potential being expressed as an infinite series of functions. The coefficients in the series are then chosen so as to satisfy all boundary conditions in the problem.

2. Procedures

1. Read Ch. 3, "Solution of Electrostatic Problems," Secs. 3 - 1 to 3 - 6 of the text. Write down Poisson's equation (3-5b) and Laplace's equation (3-9). Write down the definitions of all symbols involved in these equations. Write down the derivation of Poisson's equation beginning with the differential form of Gauss' Law (3-3).

2. Read the Supplementary Notes, Sec. I, for a discussion of spherical and cylindrical coordinates. Write down the definitions of the spherical and cylindrical coordinates, including the geometric diagrams in the Supplementary Notes. Write down the algebraic equations given in eqs. 3 and 4 of the Supplementary Notes for the relationship of both sets of coordinates to rectangular coordinates.

   Note: On the Unit test you will be given an expression for the Laplacian. You will then have to state whether or not the expression is a valid form of Laplacian and which coordinate system is used to write down the Laplacian.

3. Write down Theorem I (Superposition Theorem) and Theorem II (Uniqueness Theorem). See the Supplementary Notes, Sec. II, for a discussion of the importance of these two Theorems in solving Laplace's equation. You may be asked for such a discussion on the unit test.

4. Write down the derivation of the two ordinary differential equations (eq. 3-16 and 3-17) for the electrostatic potential in spherical coordinates in the case where the potential is independent of the azimuthal angle ϕ. You are to begin with eq. 3-13 and assume that

   $$U(\vec{r}) = U(r, \theta) = Z(r)P(\theta)$$

   In your derivation of eq. 13-17, you can leave the separation constant as k since $k = n(n + 1)$ where n is an integer can be established only after solving eq. 3-16.

5. Write down the derivation of the two ordinary differential equations (eq. 3-25) for the electrostatic potential in cylindrical coordinates in the case where the potential is independent of the z-coordinate. (Caution: This r and θ are different from those appearing in Procedure 4). The two ordinary differential equations come directly from eq. 3-25. That is, if $U(r,\theta) = Y(r)S(\theta)$, then the ordinary differential equation are:

$$r\frac{d}{dr}\left(r\frac{dY}{dr}\right) - kY = 0 \qquad (1)$$

and

$$\frac{d^2S}{d\theta^2} + kS = 0 \qquad (2)$$

6. Read the Supplementary Notes, Sec. III. Write down equations (6), (7), (8), and (9) of the Supplementary Notes and be prepared to write them down from memory when asked.

7. Read very carefully the solution of the potential of an uncharged conducting sphere placed in an *initially* uniform electric field $\vec{E}_0$ as given in Sec. 3-5 of the text. This is a prototype solution which can be used as a model for other problems. You will have to refer to this solution when carrying out Procedure No. 8.

8. Solve the following problems:

   Problems 3-1, 3-2, 3-8, 3-11, 3-12

Supplementary Notes

I. Spherical coordinates (r,θ,ϕ) as in Fig. 1.

Any point P can be located by the radius vector $\vec{r}$. The radius vector $\vec{r}$ can be expressed in terms of rectangular coordinates as

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Alternatively, the radius vector $\vec{r}$ can be expressed in terms of spherical coordinates (r,θ,ϕ) which are defined as follows:

i) the variable r is the magnitude of $\vec{r}$, that is, the distance from the origin to P.

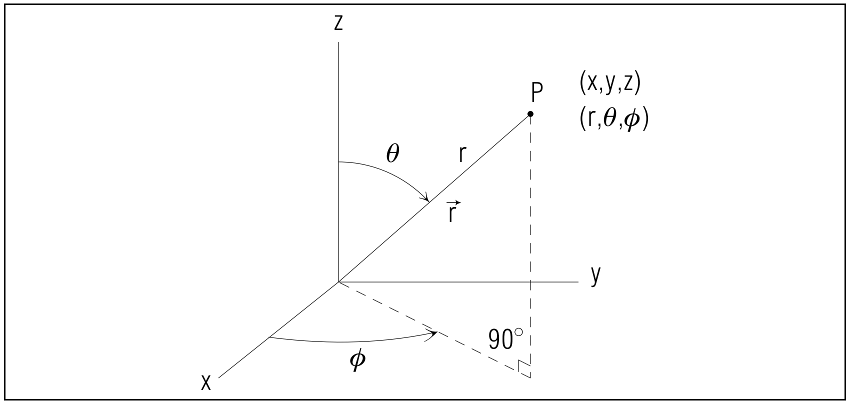

Figure 1. Spherical coordinates (r,θ,ϕ).

ii) the polar angle θ is the angle between the z-axis and $\vec{r}$.

iii) the azimuthal angle ϕ is the angle in the $x-y$ plane between the x-axis and the projection of $\vec{r}$ onto the $x-y$ plane.

It is relatively easy to derive the following algebraic relations:

$$x = r\sin\theta cos\phi \qquad (3)$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Cylindrical coordinates (r,θ,z) as in Fig. 2.

In this case, the radius vector $\vec{r}$ can be expressed in terms of cylindrical coordinates (r,θ,z) which can be defined as follows

i) the variable r is the magnitude of the *projection* of $\vec{r}$ onto the $x-y$ plane.

ii) the azimuthal angle θ is the angle in the $x-y$ plane between the x-axis and the projection of $\vec{r}$ onto the $x-y$ plane.

iii) the variable z is the same as in the rectangular coordinate system with (x,y,z).

It is relatively easy to derive the following algebraic relations:

$$x = r\cos\theta \qquad (4)$$
$$y = r\sin\theta$$
$$z = z$$

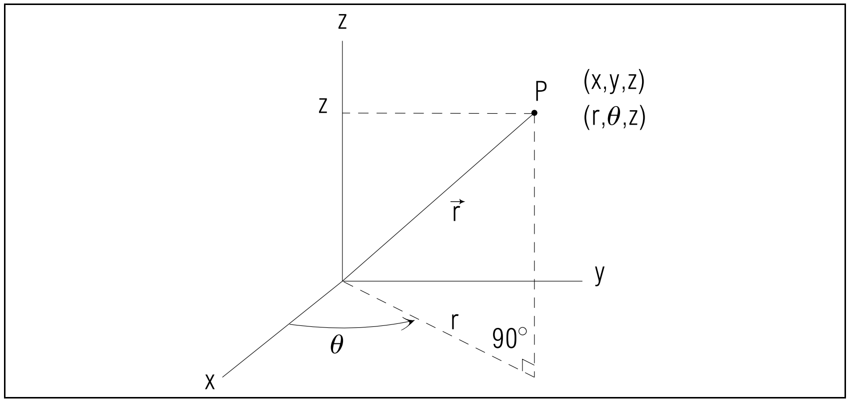

Figure 2. Cylindrical coordinates (r,θ,z).

II. Superposition and Uniqueness Theorems for Laplace's Equation

Theorem I is called the Superposition Theorem. It's importance rests with the fact that it is sometimes possible to obtain a set of functions $\{U_n(\vec{r})\}_{n=1}^{\infty}$ which are all solutions to Laplace's equation

$$\nabla^2 U_n(\vec{r}) = 0$$

These solutions can be found by the separation-of-variables method, say, in a particular coordinate system. Then, if C_n are a set of arbitrary constants, we can write

$$U(\vec{r}) = \sum_{n=1}^{\infty} C_n U_n(\vec{r}) = C_1 U_1(\vec{r}) + C_2 U_2(\vec{r}) + \dots \tag{5}$$

and obtain the result that U(n) also satisfies Laplace's equation since

$$\nabla^2 U(\vec{r}) = \nabla^2 \left(\sum_{n=1}^{\infty} C_n U_n(\vec{r}) \right) = \sum_{n=1}^{\infty} \nabla^2 U_n(\vec{r}) = 0$$

Theorem II is called the Uniqueness Theorem. It's importance rests with the fact that any solution to Laplace's equation which satisfies all boundary conditions on the solution is unique (up to an arbitrary additive constant). This means in essence that equation (5) of the Supplementary Notes is the *general* solution to Laplace's equation in the particular coordinate system. In practice, then the solution of Laplace's equation with a given set of boundary conditions proceeds as follows:

1. Choose a coordinate system which exploits any symmetries in the potential problem.
2. Write down the expansion in eq. 5 for the potential,

$$U = \sum_n C_n U_n$$

3. Use physical and / or mathematical arguments to determine the constants C_n so that U satisfies all boundary conditions. Usually, only a few of the C_n are non-zero. If the appropriate coordinate system has been chosen, this determination of the C_n is relatively straightforward.
4. The expression determined in step 3 must then be the solution to the given potential problem up to an arbitrary additive constant.

III. Zonal and Cylindrical Harmonics

The zonal and cylindrical harmonics are the solutions $U_n(\vec{r})$ in spherical and cylindrical coordinates when the potential is independent of the azimuthal coordinate ϕ and z-coordinate, respectively. The expressions for each set of solutions are as follows:

Zonal harmonics-

$$U_n = \begin{cases} r^n P_n(\theta) & n = 0,1,2,\dots \\ r^{-(n+1)} P_n(\theta) & n = 0,1,2,\dots \end{cases} \tag{6}$$

The functions $P_n(\theta)$ are known as Legendre polynomials. They are polynomials in the variable $\cos\theta$. See Table 3-1 of the text. You will need only the first few Legendre polynomials in this course. Then, the general solution to Laplace's equations is

$$U(r,\theta) = \sum_{n=0}^{\infty} A_{n+1} r^n P_n(\theta) + \sum_{n=0}^{\infty} C_{n+1} r^{-(n+1)} P_n(\theta) \tag{7}$$

The constants A_{n+1}, have been introduced to complement the C_{n+1} because of the two different types of solutions U_n. Each term in the expansion in eq. 7 can be given a physical interpretation. See Sec. 3-5 of the text for such an interpretation.

Cylindrical harmonics-

$$U_n = \begin{cases} r^n \cos n\theta, r^n \sin n\theta & n = 0,1,2,\dots \\ r^{-n} \cos n\theta, r^{-n} \sin n\theta & n = 0,1,2,\dots \end{cases} \tag{8}$$

Then,

$$U(r,\theta) = \sum_{n=0}^{\infty} A_{n+1} r^n \cos n\theta + \sum_{n=1}^{\infty} A'_{n+1} r^n \sin n\theta \tag{9}$$

$$+C_1 \ell n r + \sum_{n=1}^{\infty} C_{n+1} r^{-n} \cos n\theta + \sum_{n=1}^{\infty} C'_{n+1} r^{-n} \sin n\theta$$

It may be instructive to write out the first few terms in each sum above. Thus,

$$U(r,\theta) = (A_1 + A_2 r \cos\theta + A_3 r^2 \cos 2\theta + \ldots) \tag{10}$$

$$+(A'_2 r \sin\theta + A'_3 r^2 \sin 2\theta + \ldots) + C_1 \ell n r + (C_2 \cos\frac{\theta}{r} + C_3 \cos\frac{2\theta}{r^2} + \ldots) +$$

$$(C'_2 \sin\frac{\theta}{r} + C'_3 \sin\frac{2\theta}{r^2} + \ldots)$$

Each term in eq. 10 above can be given a physical interpretation as in the previous case. The constants are chosen so as to satisfy all boundary conditions. The method of argument proceeds as outlined in Sec. 3.5 on the use of zonal harmonics.

Acknowledgments

The author would like to thank Illinois State University for support in the construction of this lesson. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.