PHYSNET

MISN-0-504

REVIEW OF ELECTROSTATICS

Electricity and Magnetism

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

REVIEW OF ELECTROSTATICS

by

R. D. Young

Title: **Review of Electrostatics**

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Input Skills:

1. Vocabulary: conservative force, potential energy (MISN-0-21); conductivity (MISN-0-118).
2. Perform basic vector analysis (MISN-0-503).
3. Expand binomial denominators using the binomial theorem.
4. Represent tensor elements using standard matrix notation (MISN-0-491).

Output Skills (Knowledge):

K1. Vocabulary: volume charge density, electric dipole moment, surface charge density, electric monopole moment, electric quadrupole moment.

K2. State the formal expressions for the electrostatic field and potential of an arbitrary distribution of charge, and the relationship between the field and the potential.

K3. State Gauss's law in both integral and differential form. Derive each form of Gauss's law from the other by use of the divergence theorem.

K4. Derive the approximate forms (to first order in the dipole separation) of the electrostatic field and potential produced by a dipole beginning with the exact forms of each quantity.

Output Skills (Rule Application):

R1. Compute the monopole, dipole and quadrupole moments of a given charge distribution.

R2. For a given set of multipole moments, compute the approximate form of the electrostatic potential far from the charge distribution.

External Resources (Required):

1. J. Reitz, F. Milford and R. Christy, *Foundations of Electromagnetic Theory*, 4th Edition, Addison-Wesley (1993).

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New authors, reviewers and field testers are welcome.

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1. Introduction

You have already studied most of the physical concepts introduced in this unit. For example, the concepts of charge, charge distribution, Coulomb's Law, the electrostatic field $\vec{E}$, the electrostatic potential, Gauss' Law, and simple electric dipoles. You have also performed numerical calculations using all of these concepts. In this unit you will review those concepts. However, the treatment is more mathematically sophisticated than the previous treatment to which you were exposed. For example, vector analysis will be used more extensively. Also, expressions for Coulomb's Law, the electrostatic field $\vec{E}$, and the electrostatic potential will be written with respect to an arbitrary coordinate system rather than having the origin at the position where the electrostatic force, field, or potential is being calculated.

In addition, this unit generalizes the discussion of electric dipoles. This is accompanied by introducing the concept of multipole expansions. By using a multipole expansion, the electrostatic field and potential of an arbitrary charge distribution can be approximately calculated to any degree of accuracy. This requires the introduction of the concepts of the monopole moment, dipole moment, quadrupole moment, etc., of a charge distribution. The moments of a charge distribution are very important in many fields when approximate calculations are being performed.

2. Procedures

1. Read Ch. 2 of the text.

2. Write down or underline the definitions, terms, and equations called for in Output Skills K1-K2. See the Supplementary Notes.

3. Write down the details of the derivation of the differential form of Gauss' Law as given at the end of section 2-6, between eqs. 2-25 to 2-28. This proof can also run in the opposite direction by beginning with the differential form of Gauss' Law and deriving the integral form.

4. Write down the derivation of the approximate form of the electrostatic field (eq. 2-36) of a simple electric dipole to first order in the dipole separation $\vec{\ell}$ beginning with the exact form of the electrostatic field (eq. 2-32). Use the hints in section 2-8. The following formula is useful here,

$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!}x^2 \dots$$

for $|x| < 1$. So, if $|\vec{\ell}| \ll 1$, then

$$\left(1 - \frac{2(\vec{r}-\vec{r'})\cdot\vec{\ell}}{|\vec{r}-\vec{r'}|^2} + \frac{\vec{\ell}^2}{|\vec{r}-\vec{r'}|^2}\right)^{-3/2} \approx \left(1 - \frac{2(\vec{r}-\vec{r'})\cdot\vec{\ell}}{|\vec{r}-\vec{r'}|^2}\right)^{-3/2}$$

$$= 1\frac{3}{2} + \frac{2(\vec{r}-\vec{r'})\cdot\vec{\ell}}{|\vec{r}-\vec{r'}|^2} + \dots,$$

$$\left(1 - \frac{2(\vec{r}-\vec{r'})\cdot\vec{\ell}}{|\vec{r}-\vec{r'}|^2} + \frac{\vec{\ell}^2}{|\vec{r}-\vec{r'}|^2}\right)^{-3/2} \approx 1 + \frac{3(\vec{r}-\vec{r'})\cdot\vec{\ell}}{|\vec{r}-\vec{r'}|^2} + \dots$$

to first order in $\vec{\ell}$. Also,

$$\lim_{|\vec{\ell}|\to 0} \vec{p} = q\,\vec{\ell}.$$

Of course the charge q must increase as ℓ decreases, in going to the limit, so that $\vec{p}$ remains finite.

These results are used in the derivation. Write down the derivation of the approximate form of the electrostatic potential (eq. 2-39) to first order in the dipole separation $\vec{\ell}$ beginning with the exact form of the electrostatic field.

5. Write down the expression of the approximate form of the electrostatic potential (eq. 2-48) of an arbitrary charge distribution beginning with the exact form of the potential (eq. 2-45). This expression requires some comment. The charge is localized in the volume V' with charge density $\rho(\vec{r}\prime)$ as shown in Fig. 2-12.

Suppose the parameter d is a dimension characteristic of the spatial extent of the volume V'. Then the observation point is taken to be $\vec{r}$ such that $d/r \ll 1$ where $r = |\vec{r}|$. Then, the expansion of the denominator $|\vec{r} - \vec{r}\prime|^{-1}$ is made in terms of r'/r. Terms involving cube and higher powers of r'/r are dropped. This is the reasoning used between eq. 2-46 and 2-47 of the text. The final approximate result for the potential can be expressed as:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \sum_{i=1}\sum_{j=1}\frac{1}{2}\frac{x_i x_j}{r^5}Q_{ij} + \ldots\right)$$

where

$$Q = \text{ (monopole moment) } = \int_{V'} \rho(\vec{r'})dV'$$

$$\vec{p} = \text{ (dipole moments) } = \int_{V'} \vec{r}\rho(\vec{r'})dV'$$

$$Q_{ij} = \text{ (quadrupole moments) } = \int_{V'} (3x'_i x'_j - \delta_{ij} r'^2)\rho(\vec{r'})dV'$$

This type of expansion is known as a multipole expansion.[1] The approximation gets better the more terms which are included. The next higher-order term beyond the quadrupole moment term is the octopole moment term. The octopole moment is important in nuclear physics but we shall not delve into it here. Note that the monopole moment is just the total charge.

6. Solve the following problems in the text: 2-23, 2-24, 2-25. Note: to solve 2-23 start with the expression:

$$\vec{F}_{ext} = -q\vec{E}_{ext}(\vec{r}) + q\vec{E}_{ext}(\vec{r} + \vec{\ell})$$

where

$$\vec{E}_{ext}(\vec{r} + \vec{\ell}) - \vec{E}_{ext}(\vec{r}) + \vec{\ell}\cdot\vec{\nabla}\vec{E}_{ext}(\vec{r}) + \ldots$$

Also, in 2-24 do not plot the equipotential surfaces. Keep up to the quadrupole term in the potential, however.

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[1]

$$Q_{11} = \int_{V'} (3x'_1 x'_1 - \delta_{11} r'^2)dV' = \int_{V'} \rho(3x'^2_1 x'^2_1 - x'^2_2 - x'^2_3)dV'$$

$$Q_{11} = \int_{V'} \rho(2x'^2_1 - x'^2_2 - x'^2_3)dV', \text{ since } \delta_{11} = 1$$

$$Q_{12} = \int_{V'} \rho(3x'_1 x'_2 - \delta_{12} r'^2)dV' = \int_{V'} \rho 3x'_1 x'_2 dV'$$

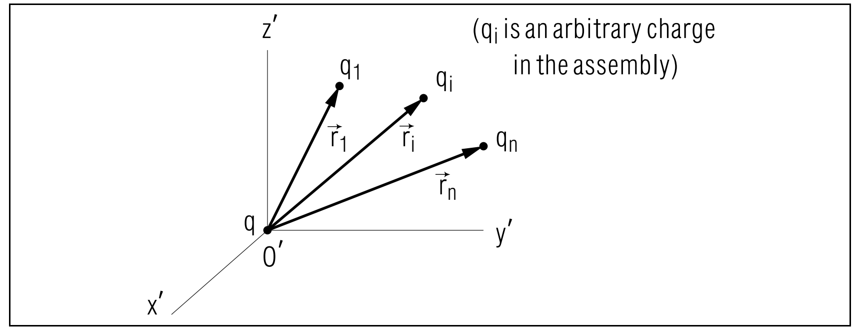

Figure 1. .

3. Supplementary Notes

Reference to an arbitrary origin

In your past experiences with electrostatics, you have written down expressions for electrostatic forces, fields, and potentials where the point of interest was at the origin of the coordinate system. For example, consider the electrostatic force on a point charge q due to an assembly of point charges $q_1, q_2, \ldots, q_n$ located at $\vec{r'}_1$, $\vec{r'}_2$, $\vec{r'}_3$, $\ldots$, $\vec{r'}_n$ respectively. The point charge q is located at the origin (see Fig. 1).

Then the force $\vec{F}$ on q is given by Coulomb's Law as,

$$F = \frac{q}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i \frac{(-\vec{r'}_i)}{|\vec{r'}_i|^3}$$

where

$$\vec{r'}_i = x'_i\vec{i} + y'_i\vec{j} + z'_i\vec{k}$$

and

$$|\vec{r'}_i| = \sqrt{x'^2_i + y'^2_i + z'^2_i}$$

The minus sign is present because the vector $\vec{r'}_i$ between q_i and q is usually drawn *from* q_i to q rather than from q to q_i as we have it here.

In the present text, the situation is generalized by allowing the charge q to be located at $\vec{r}$ relative to some coordinate system (see Fig. 2).

This arbitrary coordinate system is denoted by unprimed quantities. It should be pointed out that the vectors $\vec{r}_i$ are considered as *constant*

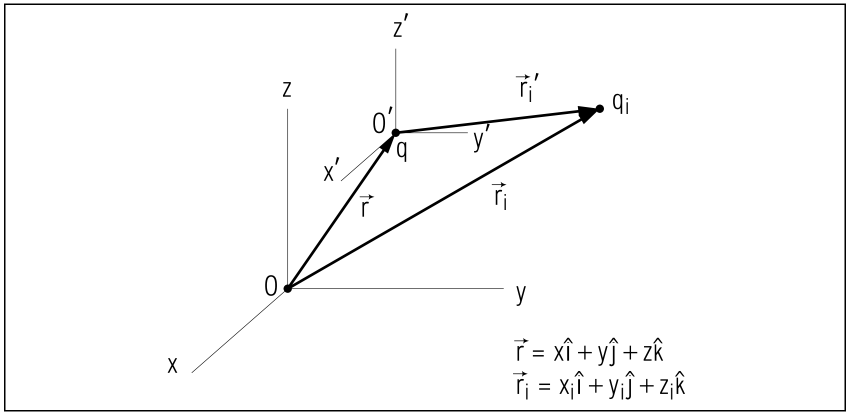

Figure 2. .

vectors (so the components are just constant scalars). This happens because the charge assembly is fixed. However, the charge q is free to move so the vector $\vec{i}$ is considered as a variable vector. It is obvious that,

$$\vec{r}_i = \vec{r} + \vec{r'}_i$$

and that,

$$-\vec{r'}_i = \vec{r} - \vec{r}_i$$

Combining (I-1) and (I-2) gives,

$$\vec{F} = \vec{F}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}|^3}$$

This is the first term in eq. (2-7) of the text. The other terms can be obtained in a similar manner. The notation in (I-4) is meant to dramatize that the force is now a *function* of the position vector $\vec{r}$ of the charge q. So,

$$\vec{F} = \vec{F}(\vec{r}) = (F_x(x,y,z)\vec{i} + F_y(x,y,z)\vec{j} + F_z(x,y,z)\vec{k}$$

and

$$F_x(x,y,z) = \frac{q}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i \frac{x - x_i}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{3/2}}$$

etc.. Keep in mind that (x, y, z) are variables above while all other symbols, including (x_i, y_i, z_i), are *constants*.

Acknowledgments

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