MISN-0-501

CENTRAL FORCE MOTION

Classical Mechanics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

CENTRAL FORCE MOTION

by

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Title: **Central Force Motion**

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Input Skills:

1. For a given mechanical system use Lagrange's equations to obtain the equations of motion for the system (MISN-0-498).

Output Skills (Knowledge):

K1. For arbitrary central force motion (a) determine the expression for the Lagrangian in terms of center of mass and relative coordinates, using the reduced mass concept. (b) Determine the first integrals of motion and the generalized form of Kepler's second law. (c) Determine the differential equation for r as a function of angle. (d) Define turning points and effective potentials and explain how turning points can be determined from a graph of the effective potential.

K2. For planetary like motion obtain the general equation of the orbit and classify the orbits in terms of conic sections as a function of the energy. Include definitions and/or explanations.

K3. Derive Kepler's third law for planetary like motion.

Output Skills (Problem Solving):

S1. Given a particular central force describe qualitatively the orbit using a graph of effective potential. Set up and solve the differential equation for the orbit.

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1. Introduction

There are two types of motion which play especially important roles in classical as well as quantum) mechanics - harmonic motion and central-force motion. Harmonic motion was covered in units 5 and 6 using Newtons laws of motion. Central-force motion has been reserved for this unit to illustrate the use of the Lagrangian formulation. All of the results of this unit can be obtained, of course, without recourse to the Lagrangian formulation. In fact you may have obtained them in a previous mechanics course. If that is the case, this unit will serve as a review of central-force motion as well as an example of the application of the Lagrangian technique.

2. Procedures

1. a. Read sections 8.1 and 8.2 of Marion.

   Exercise: Fill in the details leading to eq. 8.4 of Marion.

   b. Read section 8.3 of Marion.

   Exercise: Fill in any missing details in the analysis leading to eqs. 8.7, 8.10, 8.12 and 8.14 of Marion.

   c. Read section 8.4 of Marion.

   Exercise: Fill in any missing details in the analysis leading to eqs. 8.17 and 8.20.

   d. Read sections 8.5 and 8.6 of Marion.

2. Read section 8.7 of Marion. Note that a and b are generally called the *semi-major* and *semi-minor* axes while $2a$ and $2b$ are called the *major* and *minor* axes.

   Exercise: Fill in any missing details in the analysis leading to eq. 8.39 and 8.41.

   Review the definitions and properties of conic sections (as needed) by working through Sect. 3.

   Exercise: Fill in any missing details in the analysis leading to eqs. 8.42 - 8.44 and 8.48.

   Read section 8.9 and the first two paragraphs of section 8.10 in Marion. You are not responsible for this material, but it is worth reading through one time.

3. Work problems 8-4, 8-13, 8-14, 8-15 and 8-23 in Marion.

3. Conic Sections

A1l conic sections can be represented by the single mathematical expression

$$\frac{\alpha}{r} = 1 + \epsilon \cos\theta\,,$$

where r and θ are radial and angular variables respectively while ϵ and α are constants:

$$\epsilon = \text{ eccentricity,}$$

$$\alpha = \text{ semi-latus rectum.}$$

The name conic section comes from the observation that these curves are precisely those obtained as the perimeters of slices (or sections) through a right circular cone.

1. ellipse - the locus of points for which the sum of the distances from two fixed points (F, $F' =$ foci) is a constant (see Fig. 1).

   By definition for any point P on the ellipse:

$$r + r' = 2Q = \text{ constant.}$$

   Inspection of the figure then shows that

$$OV = FB = a = \text{ semimajor axis,}$$

   and

$$VV' = 2a = \text{ major axis } = \text{ intervertical distance.}$$

   For convenience let:

$$b = OB = \text{ semiminor axis,}$$

$$\epsilon = \text{ interfocal distance.}$$

   Then inspection of the figure again shows that

$$OF = a\epsilon\,,$$

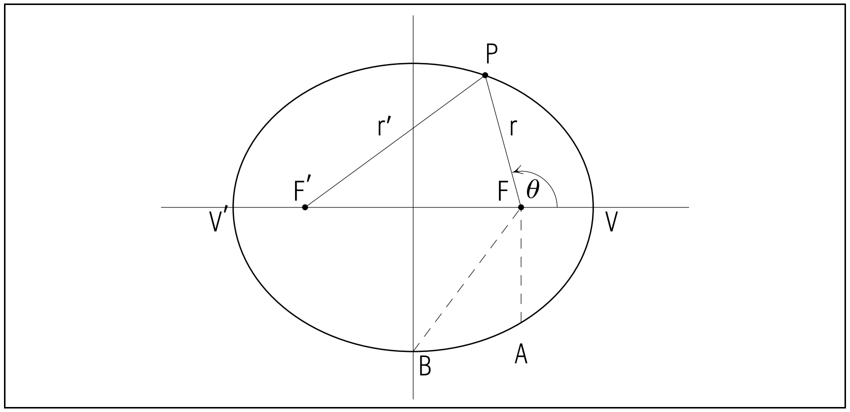

Figure 1. .

$$b = a\sqrt{1-\epsilon^2}\,.$$

To determine the equation of the ellipse note that the definition gives:

$$(r')^2 = (2a-r)^2 = 4a^2 - 4ar + r^2\,,$$

while the law of cosines gives

$$(r')^2 = r^2 + 4a^2\epsilon^2 - 2(2a\epsilon)r\cos(\pi-\theta)\,,$$

$$= r^2 + 4a^2\epsilon^2 + 4a\epsilon r\cos\theta\,.$$

Equating these and rearranging gives

$$\frac{a(1-\epsilon^2)}{r} = 1 + \epsilon\cos\theta\,.$$

This is the equation of a conic section with

$$\epsilon = \text{ eccentricity}$$

(justifying the earlier choice of notation)

$$\alpha = a(1-\epsilon^2) = \text{ semi-latus rectum}\,.$$

Letting $\theta = \pi/2$ in the equation of the ellipse and consulting the figure reveals that

$$FA = \alpha\,.$$

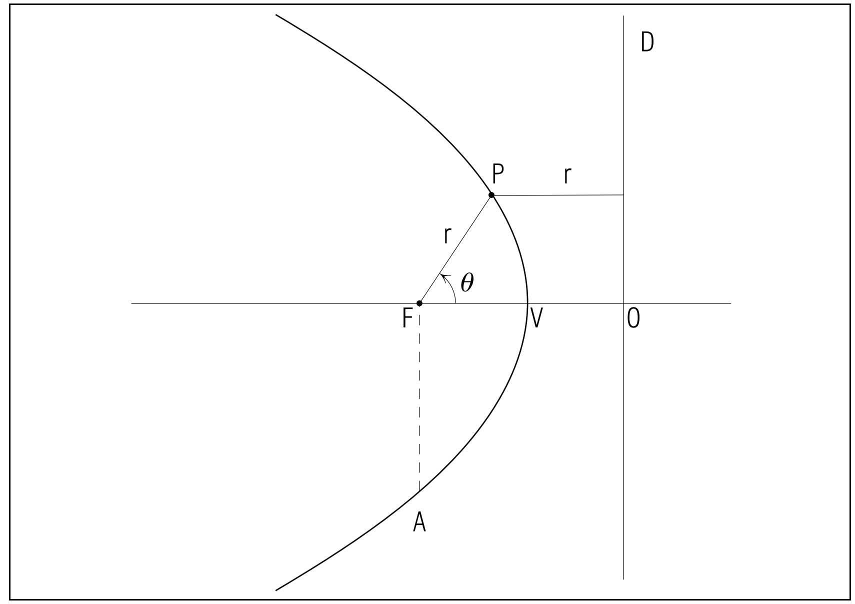

Figure 2. .

2. parabola - the locus of points equidistant from a straight line (D = directrix) and a point (F = focus) (see Fig. 2).

   For convenience let

   $$\frac{a}{2} = FV = VO\,.$$

   Then from the figure for any point P on the parabola

   $$r + r\cos\theta = a\,,$$

   so that

   $$\frac{a}{r} = 1 + \cos\theta\,.$$

   This is the equation of a conic section with an eccentricity of 1 and with

   $$\alpha = a = \text{ semi-latus rectum.}$$

   Letting $\theta = -\pi/2$, shows that

   $$FA = \alpha\,.$$

3. hyperbola - the locus of points for which the difference between the distances from two fixed points (F, F' = foci) is a constant (see Fig. 3).

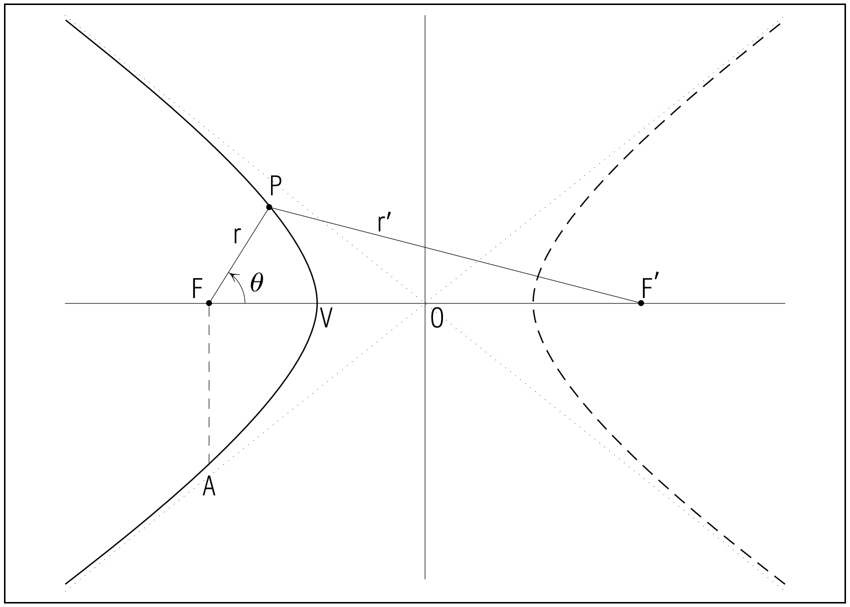

Figure 3. .

By definition for any point P on the hyperbola

$$r - r' = 2a = \text{ constant.}$$

Inspection of the figure then shows that

$$OV = a\,.$$

As in the case of the ellipse it is convenient to define

$$\epsilon = \frac{\text{interfocal distance}}{\text{intervertical distance}}\,.$$

The definition of a hyperbola then gives

$$(r')^2 = (2a + r)^2 = 4a^2 + 4ar + r^2\,,$$

which with the law of cosines gives

$$(r')^2 = r^2 + 4a^2\epsilon^2 - 4(4a\epsilon)r\cos\theta\,.$$

Equating and rearranging yields

$$\frac{a(\epsilon^2 - 1)}{r} = 1 + \epsilon\cos\theta\,.$$

Acknowledgments

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