PHYSNET

MISN-0-498

HAMILTON'S PRINCIPLE, LAGRANGE'S EQUATIONS

Classical Mechanics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

HAMILTON'S PRINCIPLE, LAGRANGE'S EQUATIONS

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Title: **Hamilton's Principle, Lagrange's Equations**

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Input Skills:

1. Use the variational technique to obtain Euler's equations for the solution of the basic problem in the calculus of variations (MISN-0-497).

Output Skills (Knowledge):

K1. Vocabulary: Lagrangian, degree of freedom, generalized coordinates (proper and improper), configuration space, phase space, holonomic constraints.

K2. State Hamilton's principle in terms of generalized coordinates.

K3. Explain the physical significance of Lagrange's undetermined multipliers.

K4. Derive Lagrange's equations for generalized coordinates from Hamilton's principle. State the conditions under which the derivation is valid.

K5. Derive Lagrange's equations with undetermined multipliers for systems with constraints. State the conditions undo which the derivation is valid.

Output Skills (Problem Solving):

S1. For a given mechanical system find the Lagrangian and use Lagrange's equations to obtain the equations of motion for the system.

S2. Use Lagrange's equations with undetermined multipliers to find the forces of constraint on a given system.

External Resources (Required):

1. J. Marion, *Classical Dynamics*, Academic Press (1988).

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1. Introduction

The previous unit established the mathematical fundamentals of variational calculus. The present unit is concerned with the application of the variational calculus to classical mechanics. In particular, it will be seen that all of classical mechanics can be formulated in terms of a single variational principle known as Hamilton's principle. Euler's equations resulting from Hamilton's principle are known as the Euler - LaGrange equations or simply Lagrange's equations. These equations will be used in this unit to solve a number of mechanical problems which are difficult to handle in the Newtonian formulation. It should be noted that the formulation of classical mechanics in terms of Hamilton's principle is completely equivalent to the Newtonian formulation. No new physics is included in Hamilton's principle! However, as pointed out in the introduction to MISN-0-497, the variational approach does provide new insights and facilitates the solution of certain kinds of problems.

2. Procedures

1. Read sections 7.1 through 7.4 of Marion. On this first reading, concentrate on the general ideas and the important definitions. Do not be concerned with mathematical details.

   Optional - Read sections 6-1 through 6-3 in Greenwood.

   Write down definitions for each of the quantities in Output Skill K1.

   Optional - Marion does not discuss generalized forces. The simplest way to define the generalized force Q. associated with the generalized coordinate q_j is as follows

   $$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \,,$$

   where $\vec{F}_i$ is the total force on the i^{th} particle and $\vec{r}_i$ is the position of the i^{th} particle. Note that the sum is over all particles. The significance

   of generalized forces can be seen by considering a small change (δq_j) in the j^{th} generalized coordinate at a particular time while holding all other generalized coordinates fixed. Then

   $$Q_j \delta q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_i \vec{F}_i \cdot \delta \vec{r}_i = \delta W \,.$$

   Thus $Q_j \delta q_j$ is the amount of work that would be done by all forces acting at the chosen time if the system were to undergo a displacement in which the j^{th} generalized coordinate alone changed by an amount δq_j. The similarity between $Q_j \delta q_j$ for generalized quantities and $F_x \delta_x$ for cartesian quantities makes the usage of the terminology "generalized force" for Q_j self-evident.

   Note: If the generalized coordinates are improper then the constraint equations are ignored when the displacements δq_j are made.

   Optional - Read section 6-5 in Greenwood.

2. Reread sections 7.1 through 7.4 in Marion. Fill in mathematical details where necessary.

   Write down from memory Hamilton's principle in terms of generalized coordinates.

   Starting from Hamilton's principle in terms of generalized coordinates, derive Lagrange's equations. Actually, carry through the variational calculus. Don't simply make a correspondence as is done in the text.

   Write down the two conditions under which the preceding derivation is valid.

   Note: For non-conservative systems, there is a generalized version of Hamilton's principle that leads to the Euler equations

   $$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \,.$$

   This generalization will not be considered in this course. (The interested student is referred to Goldstein, pp 38 - 40)

3. (Optional)
   Read section 7.5 in Marion. Unfortunately, this presentation is not completely clear. Therefore, it is strongly recommended that you also read Goldstein, pp. 40 - 44 (Begin at the second paragraph on p. 40).

   Marion states near the bottom of p. 207 that the Lagrange multipliers are the forces of constraint. It is clear from the discussion in Goldstein

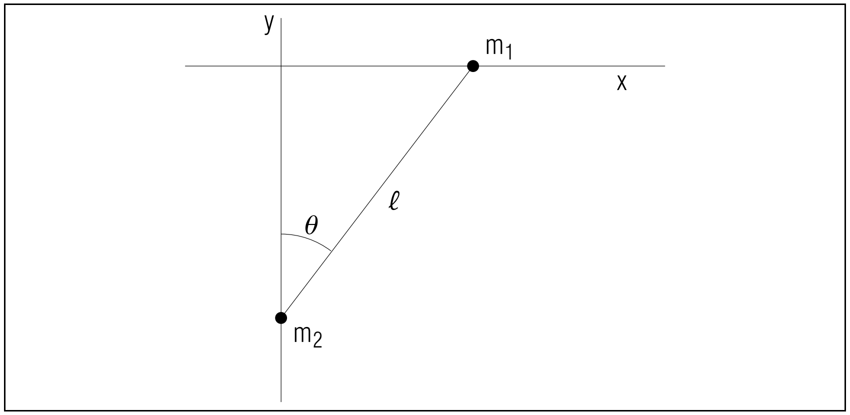

Figure 1. .

that this is not true in general. The only thing that one can say in general is that:

$$\sum_{\ell} \lambda_\ell a_{\ell k} = Q'_k = k^{\text{th}} \text{ generalized force of constraint}\,.$$

In Example 7.9 in Marion (essentially the same as Goldstein's example on p. 43) it is fortuitous that the Lagrange multiplier is precisely the frictional force. Another choice of generalized coordinates would have resulted in a less transparent interpretation of the Lagrange multiplier.

4. Read sections 7.3 and 7.7 in Marion.

▷ Work these problems in Marion: 7-3, 7-4, 7-9

Optional, for Output Skill K5:

▷ Problem – Particles m_1 and m_2 are constrained by frictionless constraints to move in the vertical x-y plane such that m_1 remains on the horizontal x-axis and m_2, remains on the vertical y-axis (see Fig. 1). The particles are connected by a massless inextensible string of length ℓ. The initial conditions on m_1 are $x(0) = \ell$, $\dot{x}(0) = 0$ and the corresponding conditions on m_2 are $y(0) = 0$, $\dot{y}(0) = 0$. For the case where $m_1 = m_2 = 0$, solve for the tension P in the string as a function of the angle θ. What is the period of motion in θ?

Acknowledgments

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