PHYSNET

MISN-0-490

CAUCHY RESIDUE THEOREM AND DEFINITE INTEGRALS

Math
Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

CAUCHY RESIDUE THEOREM AND DEFINITE INTEGRALS

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Title: **Cauchy Residue Theorem and Definite Integrals**

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Version: 2/1/2000 Evaluation: Stage B0

Length: 2 hr; 9 pages

Input Skills:

1. Vocabulary: conformal mapping, complex variables, singularities, residues, mapping, transformations: translation, rotation, stretching, inversion, linear.
2. Unknown: assume (MISN-0-489).

Output Skills (Knowledge):

K1. Write the definition or explain the meaning of the Cauchy principle value of an integral.

K2. Write down the Cauchy residue theorem when asked, and include the proper conditions on the functions involved.

Output Skills (Rule Application):

R1. Evaluate various definite integrals using the Cauchy residue theorem.

External Resources (Required):

1. G. Arfken, Mathematical Methods for Physicist, Academic Press (1995).
2. Schaum's Outline: Murray Spiegel, *Theory and Problems of Advanced Mathematics for Scientists and Engineers*, McGraw-Hill Book Co. (1971).

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1. Introduction

This is the last unit in your introduction to the theory of complex variables and its applications to physics and engineering. The Cauchy Residue Theorem is of powerful use in evaluating many of the integrals which regularly occur in physics and enginering.

2. Procedures

1. Review Procedure - Read Sec. 7.1, Singularities.
2. Read Sec. 7.2, Calculus of Residues in Arfken.
3. Read these sections in Spiegel:

   Residue Theorem, pages 289 to 290

   Evaluation of Definite Integrals, page 290
4. Write down the definition of Cauchy Principle Value as given in eq. (7.12) of Arfken.
5. Write down the Cauchy Integral Theorem 13.1 on page 290 of Spiegel. Note all conditions of $f(z)$.
6. Write down the formulas for evaluating these definite integrals:

   a.
   $$\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta$$
   as in eqns (7.25) and (7.28) of Arfken.

   b.
   $$\int_{-\infty}^{\infty} f(x)\,dx$$
   as in eq. (7.31) of Arfken. The conditions on $f(x)$ are listed immediately after eq. (7.29) of Arfken.

   c.
   $$\int_{-\infty}^{\infty} f(x)e^{\imath\alpha x}\,dx$$
   as in eq. (7.44) of Arfken. The conditions on $f(x)$ are given in eq. (7.38) of Arfken.

   d. Integrals with a singularity on the contour of integration as in Example (7.2.3) of Arfken.

   Note: These same integrals are discussed on page 290 of Spiegel as well as in the Solved Problems in Spiegel. The next procedure will refer you to these Solved Problems.
7. Read through the following Solved Problems of Spiegel:

   13.23 (Proof of Cauchy Residue Theorem)

   13.25 (Proof of Cauchy Residue Theorem)

   13.29 (Evaluation of Definite Integrals)

   13.30 (Evaluation of Definite Integrals)

   13.31 (Evaluation of Definite Integrals)

   13.32 (Evaluation of Definite Integrals)

   13.34 (Evaluation of Definite Integrals)

   13.35 (Evaluation of Definite Integrals)

   13.37 (Evaluation of Definite Integrals)
8. Solve the following problems in Arfken:

   7.2.8 (Evaluation of Definite Integrals)

   7.2.9 (Evaluation of Definite Integrals)

   7.2.12 (Evaluation of Definite Integrals)

   7.2.14 (Evaluation of Definite Integrals)
9. Solve the following problems in the Supplementary Problem section of Spiegel:

   13.88 (Evaluation of Definite Integrals)

   13.90 (Evaluation of Definite Integrals)

   13.91 (Evaluation of Definite Integrals)

   13.100 (Evaluation of Definite Integrals)

   13.103 (Evaluation of Definite Integrals)

3. Supplementary Note

At various times, the residues of the function $f(z) = 1/z^n + b$ at each of the poles are needed.

It is easy to calculate the residue using L'Hospital's rule. Thus, if z_0 is a pole of $f(z)$, then,

$$a_{-1} = \lim_{z \to z_0} \frac{(z - z_0)}{z^n + b} = \lim_{z \to z_0} \frac{d/dz(z - z_0)}{d/dz(z^n + b)}$$

$$a_{-1} = \lim_{z \to z_0} \frac{1}{nz^{n-1}} = \frac{1}{nz_0^{n-1}} = -\frac{z_0}{nb}$$

Since $z_0^n = -b$ so that $1/z_0^{n-1} = -z_0/b$.

Acknowledgments

The author would like to thank Illinois State University for support in the construction of this lesson. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.