PHYSNET

MISN-0-485

FOURIER INTEGRALS: PART II

Math Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

FOURIER INTEGRALS: PART II

by

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Title: **Fourier Integrals: Part II**

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Input Skills:

1. Vocabulary: Fourier integral expansion of $f(x)$, Fourier transform of $f(x)$, Fourier cosine transform and inverse cosine transform, Fourier integral theorem.
2. Unknown: assume (MISN-0-484).

Output Skills (Knowledge):

K1. Define or explain each of the terms and concepts as follows: Parseval's identities for Fourier transforms, convolution of $f(x)$ and $g(x)$, convolution theorem for Fourier transforms, Dirac delta function.

Output Skills (Rule Application):

R1. Evaluate infinite integrals using Parseval's identity for Fourier transforms.

R2. Solve integral equations using the convolution theorem.

R3. Solve problems involving the Dirac delta function.

External Resources (Required):

1. G. Arfken, Mathematical Methods for Physicist, Academic Press (1995).
2. Schaum's Outline: Murray Spiegel, *Theory and Problems of Advanced Mathematics for Scientists and Engineers*, McGraw-Hill Book Co. (1971).

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9/29/99, pss; 1/4/2000, pss

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1. Introduction

This unit is a continuation of the preceding unit. The new topics include Parseval's identity for Fourier transforms, the convolution theorem, and the Dirac delta function.

2. Procedures

1. Read section 15.5 of Arfken. Read pages 202-203 of Spiegel.
2. Underline in the text or write out the definitions and explanations of the terms and concepts of Output Skill K1 using an explanatory equation if necessary. One or two sentences should be sufficient.
3. Read part I of the Supplementary Notes, on the Convolution Theorem.
4. Read these Solved Problems in Spiegel:

   8.7 (Evaluation of infinite integrals using Parseval's Identity)

   8.9 (Solving integral equations using the convolution theorem)

5. Solve these Supplementary Problems in Spiegel:

   8.23 (Evaluation of infinite integrals using Parseval's Identity: Use the result of problem 8.16.)

   8.25 (Convolution Theorem: Use the result of problem 8.16 and possibly the trig identity relating $\sin^2$ to cos.)

   Solve this problem in Arfken:

   15.5.1 (Convolution Theorem)

6. Read part II of the Supplementary Notes, on the Dirac delta function. Solve the problems at the end of the Supplementary Notes.

3. Supplementary Notes: I

The Convolution Theorem

3a. The Theorem. The convolution theorem is stated in Spiegel, page 203, for arbitrary functions as long as all of the integrals exist. If the functions involved have certain symmetry properties the convolution theorem takes on alternative forms. Two examples are included in Problem 15.5.1 of Arfken which you are asked to prove. Thus:

1) If f and g are both odd functions, the convolution theorem reads

$$\frac{1}{2}\int_{-\infty}^{\infty} f(u)g(x-u)\,du = -\int_0^{\infty} F_s(\alpha)G_s(\alpha)\cos\alpha x\,d\alpha \tag{1}$$

which is the first result of Problem 15.5.1 in Arfken. The convolution of f and g $[h(x) = f * g]$, on the other hand, must be an *even* function. To see this consider

$$h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)g(x-u)\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(-u)g(-x+u)\,du$$

$$= -\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(u)g(-x-u)\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)g(-x-u)\,du$$

so $h(x) = +h(-x)$.

Thus

$$\frac{1}{2}\sqrt{2\pi}h(x) = -\int_0^{\infty} F_s(\alpha)G_s(\alpha)\cos\alpha x\,d\alpha$$

and

$$\frac{\pi}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} h(x)\cos\alpha' x\,dx =$$

$$-\int_0^{\infty} d\alpha F_s(\alpha)G_s(\alpha)\int_0^{\infty} dx\cos\alpha' x\cos\alpha x\,. \tag{2}$$

But it can be shown that

$$\frac{2}{\pi}\int_0^{\infty}\cos\alpha x\cos\alpha' x\,dx = \delta(\alpha-\alpha') \tag{3}$$

where $\delta(\alpha-\alpha')$ is the Dirac delta function. (See the next section of the Supplementary Notes. Then, try to prove Eq. 3). Thus, the alternative to Eq. 18 of Spiegel results from Eqs. 2 and 3:

$$H_c(\alpha) = -F_s(\alpha)G_s(\alpha)\,. \tag{4}$$

2) If f and g are both even functions, the convolution theorem reads

$$\frac{1}{2}\int_{-\infty}^{\infty} f(u)g(x-u)\,du = \int_0^{\infty} F_c(\alpha)G_c(\alpha)\cos\alpha x\,d\alpha \tag{5}$$

which is the second result of Problem 15.5.1 in Arfken. Again the convolution of f and g ($h = f*g$) is an *even* function. Using the same analysis as above gives

$$H_c(\alpha) = F_c(\alpha)G_C(\alpha) \tag{6}$$

as the alternative to Eq. 18 of Spiegel.

3b. Example. The solution of Solved Problem 8.9 of Spiegel is actually not quite correct since Fourier cosine transforms must be used. A proper solution rests on the use of Eqs. 5 and 6 above. To carry out this correct solution rewrite the integral equations as

$$\int_{-\infty}^{\infty}\frac{f(u)\,du}{(u-x)^2+a^2} = \frac{1}{x^2+b^2},\quad 0<a<b\,.$$

The right-hand side is an even function of x. Therefore, the left-hand side must be an even function of x. Thus can only be true if $f(u)$ is even since

$$\int_{\infty}^{-\infty}\frac{f(u)\,du}{(u+x)^2+a^2} = \int_{-\infty}^{\infty}\frac{f(-u)\,du}{(u+x)^2+a^2} =$$

$$-\int_{\infty}^{-\infty}\frac{f(u)\,du}{(u-x)^2+a^2} = \int_{-\infty}^{\infty}\frac{f(u)\,du}{(u-x)^2+a^2}\,.$$

Thus, if $g(u) = 1/(u^2+a^2)$ and the convolution of f and g (defined as $h = f*g$) is given by

$$h(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{x^2+b^2}\,,$$

the Convolution Theorem for these functions reads:

$$H_c(\alpha) = F_c(\alpha)G_c(\alpha)\,.$$

Now, from the definition of the functions,

$$H_c(\alpha) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{\cos\alpha x}{x^2+b^2}\,dx = \frac{1}{\pi b}\int_0^{\infty}\frac{\cos\alpha b x}{x^2+1}\,dx$$

$$H_c(\alpha) = \frac{1}{2b}e^{-\alpha b}\,.$$

Likewise,

$$G_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\cos\alpha x}{x^2+b^2}\,dx = \sqrt{\frac{2}{\pi}}\frac{1}{a}\int_0^{\infty}\frac{\cos\alpha x}{x^2+1}\,dx$$

$$G_c(\alpha) = \sqrt{\frac{\pi}{2}}\frac{1}{a}e^{-\alpha a}\,.$$

Thus

$$F_c(\alpha) = H_c(\alpha)/G_c(\alpha) = \frac{1}{\sqrt{2\pi}}\frac{a}{b}e^{-\alpha(b-a)}\,.$$

Now

$$\begin{aligned} f(u) &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(\alpha)\cos\alpha u\,d\alpha \\ &= \frac{1}{\pi}\frac{a}{b}\int_0^{\infty} e^{-\alpha(b-a)}\cos\alpha u\,d\alpha \\ &= \frac{a}{b}\frac{(b-a)}{u^2+(b-a)^2}\,. \end{aligned}$$

4. Supplementary Notes: II

The Dirac delta function

4a. The Function. Consider eq. (4) on page 201 of Spiegel or eq. (15.20) of Arfken,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}d\omega f(x')e^{-\imath\omega(x-x')}$$

$$= \int_{-\infty}^{\infty}dx'\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{-\imath\omega(x-x')}\right]f(x')$$

$$= \int_{-\infty}^{\infty}dx' K(x-x')f(x')\,.$$

I have purposefully written this equation in a way which shows that f(x) can be written as an integral over all x with $f(x)$ appearing in the integrand. Thus, the function multiplying $f(x)$, that is $K(x-x')$, in the integrand must have some rather strange properties. $K(x-x')$ must be very small for $x \neq x'$ so that contributions to the integrand from this portion of the integration region are small while $K(x-x')$ must be large when $x = x'$ so that contributions to the integral from the integrand when $x = x'$ are enhanced. The function $K(x-x')$ is given a special name and notation. $K(x-x')$ is written $\delta(x-x')$ and called the Dirac delta function. The dirac delta function has these two properties, by definition:

1. $\delta(x) = 0$ if $x \neq 0$.
2. $\int_a^b \delta(x)\,dx = 1$ when $a < x < b$.

Obviously, $\delta(x-x')$ satisfies these properties:

1. $\delta(x-x') = 0$ if $x \neq x'$.
2. $\int_{a'}^{b'} \delta(x-x')\,dx' = 1$ when $a' < x' < b'$.

The proof is accomplished by letting $x \to x - x'$ so that $dx \to dx'$ and $a' = a + x'$ and $b' = b + x'$.

4b. Representations of $\delta(x)$. A representation of the Dirac delta function is an explicit algebraic formula which satisfies the two criteria above. Such representations include some type of limiting process either explicitly or implicitly.

A representation of the Dirac delta function is

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath\omega x} d\omega\,.$$

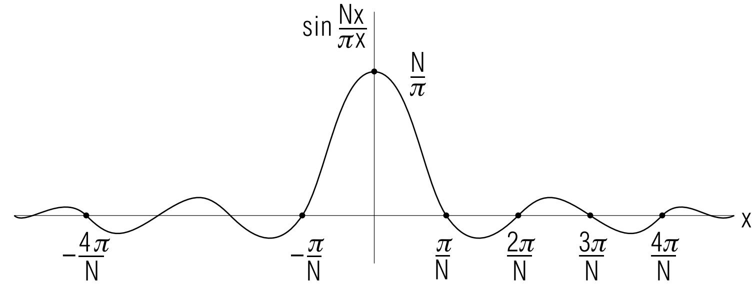
Suppose $x \neq 0$, then

$$\delta(x) = \frac{1}{2\pi}\lim_{N\to\infty}\int_{-N}^{N} e^{-\imath\omega x}\,d\omega$$

$$= \frac{1}{2\pi}\lim_{N\to\infty}\left.\frac{e^{-\imath\omega x}}{-\imath x}\right|_{-N}^{N}$$

$$= \frac{1}{\pi}\lim_{N\to\infty}\frac{\sin Nx}{x}\,.$$

Here is a graph of $\sin Nx/\pi x$:



The zeroes occur at xn/N while the function has a maximum value N/π at $x = 0$. The function approaches zero as $x \to \pm\infty$.

As $N \to \infty$, the height of the central peak goes to infinity, and the zeroes of the function approach $x = 0$ and get infinitesimally close together.

Thus

$$\lim_{N\to\infty}\frac{\sin Nx}{\pi x}$$

$= \infty$ if $x = 0$;

= a rapidly oscillating quantity (if $x \neq 0$) which when multiplied by a smooth function and integrated gives zero.

Also,

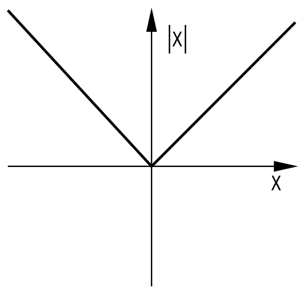
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\omega e^{-\imath\omega x} = \lim_{N\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin Nx}{x}\,dx$$

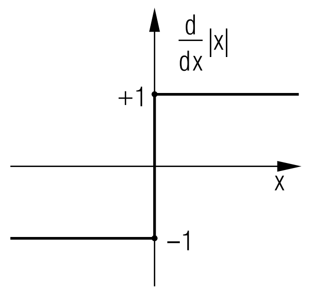
$$= \lim_{N\to\infty}\frac{1}{\pi}\cdot\pi = 1\,.$$

The interchange of the limiting process and the integration although not valid if the integral is an ordinary Riemann integral is allowed when the Dirac delta function is involved. It is obvious then that the Dirac delta function is not a function in the ordinary sense as used in introductory calculus. It is in fact a more general quantity, that is, a distribution. The mathematical step of interchanging the limit and integration above can be justified by appealing to the theory of distributions as developed by Laurent Schwartz. Essentially, the idea is to ignore the infinite jump discontinuity in $\delta(x)$ at $x = 0$ when performing calculations. For example, consider this alternative representation of the Dirac delta function:

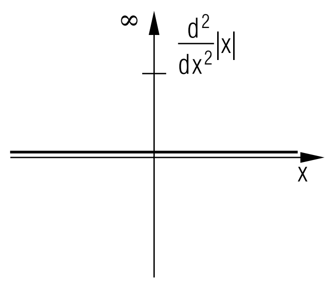
$$\delta(x) = \frac{1}{2}\frac{d^2}{dx^2}|x|, \quad -\infty < x < +\infty \,.$$

Examine these diagrams:





and



where I have separated the 0 value of $d^2/\,dx^2|x|$ for $x \neq 0$ from the x-axis for clarity. Obviously,

$$\frac{1}{2}\frac{d^2}{dx^2}|x| = 0 \text{ if } \neq 0.$$

Also,

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{d^2}{dx^2}|x|\,dx = \frac{1}{2}\lim_{N\to\infty}\left[\frac{d}{dx}|x|\right]_{-N}^{N}$$

$$= \frac{1}{2}\lim_{N\to\infty}[1-(-1)] = \lim_{N\to\infty}1 = 1\,.$$

The infinite jump discontinuity at $x = 0$ is ignored when performing the integral.

4c. Properties of the Dirac delta function.

1) $x\delta(x) = 0$ for $-\infty < x < +\infty$.

Proof:

$x\delta(x) = 0$ if $x \neq 0$ since $\delta(x) = 0$ if $x \neq 0$.

$x\delta(x) = 0$ if $x = 0$ since $0 \cdot \delta(0) = 0$ ignoring the infinite jump discontinuity at $x = 0$ in $\delta(x)$.

2) $f(x)\delta(x-a) = f(a)\delta(x-a)$

Proof:

$f(x)\delta(x-a) = [f(x) - f(a)]\delta(x-a) + f(a)\delta(x-a)$.

But, $\delta(x-a) = 0$ if $x \neq a$, and $[(f(x) - f(a)] = 0$ if $x = a$. Thus,

$$f(x)\delta(x-a) = f(a)\delta(x-a)\,.$$

Note:

$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)\int_{-\infty}^{\infty}\delta(x-a)\,dx = f(a)$.

3) $\int_{-\infty}^{\infty} f(x)\delta'(x-a)\,dx = -f'(a)$
where the prime means differentiation with respect to x.

Proof:

$\dfrac{d}{dx}[f(x)\delta(x-a)] = f'(x)\delta(x-a) + f(x)\delta'(x-a)\,.$
So,

$\dfrac{d}{dx}[f(x)\delta(x-a)] = f'(a)\delta(x-a) + f(x)\delta'(x-a)\,.$

Integrating,

$\int_{-\infty}^{\infty}\dfrac{d}{dx}[f(x)\delta(x-a)]\,dx = \int_{-\infty}^{\infty} f'(a)\delta(x-a)\,dx + \int_{-\infty}^{\infty} f(x)\delta'(x-a)\,dx,$
but

$\dfrac{d}{dx}[f(x)\delta(x-a)] = \lim_{N\to\infty}[(f(x)\delta(x-a)]_{-N}^{N} = \lim_{N\to\infty}0 = 0\,.$
So,

$0 = f'(a) + \int_{-\infty}^{\infty} f(x)\delta'(x-a)\,dx\,.$

4) $\delta(x) = +\delta(-x)\,.$

Proof:

$\delta(x) = \delta(-x) = 0$ if $x \neq 0$.

Also,

$\int_{-\infty}^{\infty} \delta(-x)\,dx = -\int_{\infty}^{-\infty} \delta(+x)\,dx = \int_{-\infty}^{\infty} \delta(x)\,dx = 1\,.$

4d. Problems on the Dirac delta function.

1) Prove that if a is real and $a \neq 0$, then

$$\delta(ax) = \frac{1}{|a|}\delta(x)\,.$$

2) Show that:

$$\delta[(x-a)(x-b)] = \frac{1}{|a-b|}[\delta(x-a) + \delta(x-b)]\,.$$

3) Show that each of these is an alternative representation of the Dirac delta function:

$$\delta(x) = \frac{1}{\sqrt{\pi}}\lim_{\epsilon\to 0}\frac{1}{\sqrt{\epsilon}}e^{-x^2/\epsilon}$$

$$\delta(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2}\,.$$

4d. Supplemental Reading. (Optional) For more discussion on the Dirac delta function, read section 1.15 of Arfken.

Acknowledgments

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