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FOURIER ANALYSIS - SERIES: PART II

Math Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

FOURIER ANALYSIS - SERIES: PART II

by

R. D. Young, Dept. of Physics, Illinois State Univ.

Title: **Fourier Analysis - Series: Part II**

Author: R. D. Young, Dept. of Physics, Illinois State Univ.

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Input Skills:

1. Vocabulary: periodic function, trigonometric function, sine, cosine, sound waves, electromagnetic waves, Taylor series.
2. Unknown: assume (MISN-0-482).

Output Skills (Knowledge):

K1. Write a definition or explain the following terms: Parseval's identity for Fourier series, set of orthogonal functions, normalized functions, normalized coefficients, set of orthonormal functions, Kronecker delta function, orthonormal series, density or weight function, complete set of functions.

Output Skills (Rule Application):

R1. Write down Fourier series in complex notation. Compute Fourier coefficients c_n in complex notation.

R2. Use Parseval's identity and Fourier series to evaluate limits of series numerically.

R3. Test whether a given set of functions is orthogonal and compute normalization coefficients for each function.

External Resources (Required):

1. G. Arfken, Mathematical Methods for Physicist, Academic Press (1995).
2. Schaum's Outline: Murray Spiegel, *Theory and Problems of Advanced Mathematics for Scientists and Engineers*, McGraw-Hill Book Co. (1971).

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1. Introduction

This unit is a continuation of the material which was started in the preceding unit. The new topics include integration and differentiation of Fourier series, Parseval's identity, complex Fourier series, and orthonormal functions and series.

2. Procedures

1. Read from the middle of page 184 to 185 of Spiegel. Read section 14.4 of Arfken.

2. Underline in the text or write out the definitions and explanations of the terms and concepts of Output Skill K1 using an explanatory equation where necessary. One or two sentences should be sufficient.

3. Memorize Theorem 7-2 of Spiegel.

4. Memorize and write from memory Theorem 1-11 on page 7 of Spiegel. This theorem gives the conditions for differentiating a Fourier series term-by-term and having the resultant series converge to the derivative of the function. See the Supplementary Notes, item 3.

5. Read the Supplementary Notes.

6. Read these Solved Problems in Spiegel;

   7.13 (Parseval's identity)
   7.14 (Differentiation and Integration)
   7.24 (Orthogonal functions)

7. Solve these Supplementary Problems in Spiegel:

   7.37 (Refer to 7.32b not 7.36b; differentiation)
   7.38 (Parseval's identity)*
   7.44 (Orthogonal functions)
   7.49 (Orthogonal functions)

* Replace 7.38b by

$$\sum_{n+1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}\,.$$

Solve Problem 14.4.1 (Integration) in Arfken.

8. Read through this following example for computing the Fourier coefficients c_n in complex notation:

Let $f(x) = 3x$ on $-\pi \le x \le \pi$. Compute the Fourier coefficients c_n where

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\imath nx}$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\imath nx}\,dx\,.$$

Thus

$$c_n = \frac{3}{2\pi}\int_{-\pi}^{\pi} xe^{-\imath nx}\,dx\,.$$

From integral tables,

$$c_n = \frac{3}{2\pi}\left[\frac{e^{-\imath nx}}{-n^2}(-\imath nx - 1)\right]_{x=-\pi}^{x=\pi}$$

and

$$c_n = \frac{3\imath(-1)^n}{n},\ n \ne 0$$

from $n = 0$,

$$c_n = \frac{3}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0\,.$$

Thus

$$f(x) = 3\imath\sum_{-\infty}^{\infty}\frac{(-1)^n}{n}e^{\imath nx},\ n \ne 0\,.$$

Of course, the Fourier coefficients can can be found by computing a_n and b_n and then using the formulas in the Supplementary Notes to calculate c_n.

Problem: Compute the Fourier coefficients c_n using the method outlined in this procedure for these two cases:

i) $f(x) = 1,\ \ -\pi \le x \le \pi$.
ii) $f(x) = x^2,\ \ -\pi \le x \le \pi$.

3. Supplementary Notes

3a. Complex notation and Fourier series. Suppose $f(x)$ satisfied Dirichlet conditions on $(-L, L)$. Then, at a point of continuity,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,\ n \geq 1$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx,\ n \geq 1\,.$$

By definition

$$\cos\phi = \frac{e^{\imath\phi} + e^{-\imath\phi}}{2}$$

and

$$\sin\phi = \frac{e^{\imath\phi} - e^{-\imath\phi}}{2\imath}\,.$$

Then

$$f(x) = \frac{a_0}{2} + \sum_n\left[\frac{1}{2}(a_n - \imath b_n)e^{\frac{\imath n\pi x}{L}} + \frac{1}{2}(a_n + \imath b_n)e^{-\frac{\imath n\pi x}{L}}\right]$$

$$= \sum_{n=1}^{\infty}\left(\frac{a_n + \imath b_n}{2}\right)e^{-\frac{\imath n\pi x}{L}} + \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - \imath b_n}{2}\right)e^{\frac{\imath n\pi x}{L}}\,.$$

Let n be replaced by $-n$ in the first summation. So:

$$f(x) = \sum_{n=-\infty}^{-1}\left(\frac{a_{-n} + \imath b_{-n}}{2}\right)e^{\frac{\imath n\pi x}{L}} + \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - \imath b_n}{2}\right)e^{\frac{\imath n\pi x}{L}}\,.$$

Define

$$c_n = \frac{a_{-n} + \imath b_{-n}}{2} \text{ if } n < 0$$

$$c_0 = \frac{a_0}{2}$$

and

$$c_n = \frac{a_n - \imath b_n}{2} \text{ if } n > 0\,.$$

Then

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{\imath n\pi x}{L}}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{\imath n\pi x}{L}}\,dx\,.$$

3b. Orthogonal functions. Equations (9), (10), (11), (12), (13), and (15) of Chapter 7 in Spiegel are only true if the functions involved are real. If the functions are complex as in Supplementary Problem 7.44, then these equations need to be changed. The change to be made is simply to include complex conjugation for one of the functions. So, the more general equations are:

9':

$$\int_a^b A^*(x)B(x)\,dx = 0$$

10':

$$\int_a^b A^*(x)A(x)\,dx = \int_a^b |A(x)|^2\,dx = 1$$

11':

$$\int_a^b \phi_m^*(x)\phi_n(x)\,dx = 0,\ \ m \neq n$$

12':

$$\int_a^b \phi_m^*(x)\phi_n(x)\,dx = \int_a^b |\phi_m(x)|^2\,dx = 1$$

13':

$$\int_a^b \phi_m^*(x)\phi_n(x)\,dx = \delta_{mn}$$

and

15':

$$\int_a^b \omega(x)\psi_m^*(x)\psi_n(x)\,dx = \delta_{mn}\,.$$

By definition, the process of complex conjugation is given as

$$(a + \imath b)^* = a - \imath b$$

where a and b are real and $\imath = \sqrt{-1}$.

3c. Conditions for differentiating term-by-term. As mentioned in Procedure 5, the conditions for differentiating a Fourier series term-by-term and having the resultant series converge to the derivative of the limit of the original series are listed in Theorem 1-11 on page 7 of Spiegel. This theorem can be written in notation relevant to this Unit as follows:

Theorem 1-11. If $f(x)$ satisfied Dirichlet conditions on $(-L, L)$ and is continuous at x so that the series

$$\frac{a_0}{2} + \sum_{n=0}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

converges to $f(x)$ while the series

$$\frac{\pi}{L}\sum_{n=0}^{\infty} n\left(a_n \sin\frac{n\pi x}{L} - b_n \cos\frac{n\pi x}{L}\right)$$

is *uniformly* convergent in $(-L, L)$, then

$$\frac{df}{dx} = \frac{\pi}{L}\sum_{n=0}^{\infty} n\left(a_n \sin\frac{n\pi x}{L} - b_n \cos\frac{n\pi x}{L}\right) .$$

3d. Normalization coefficient. Given a function $\psi(x)$ on $a \le x \le b$. The constant C is called the *normalization coefficient* of $\psi(x)$ on $a \le x \le b$ if

$$\psi'(x) = C\psi(x)$$

$$\int_a^b |\psi'(x)|^2\, dx = 1\, .$$

3e. Complete set of functions. Consider a set of functions $\{\psi_n(x)\}_{n=1}^{N}$ defined on $a \le x \le b$ such that the $\psi_n(x)$ are an orthonormal set with weight function $\omega(x)$. That is,

$$\int_a^b \omega(x)\psi_m^*(x)\psi_n(x)\, dx = \delta_{mn}\, .$$

This set of functions is a *complete set* if no non-trivial function $g(x)$ exists such that

$$\int_a^b \omega(x)\psi_n^*(x)g(x)\, dx = 0$$

for all n. Then, we can expand an arbitrary function $f(x)$ defined on $a \le x \le b$ and satisfying certain continuity criterion as

$$f(x) = \sum_n c_n\psi_n(x)$$

where

$$c_n = \int_a^b \omega(x)\psi_n^*(x)f(x)\, dx\, .$$

Examples:

(a) $\left\{\sqrt{\frac{1}{2L}}, \sqrt{\frac{1}{L}}\sin\frac{n\pi x}{L}, \sqrt{\frac{1}{L}}\cos\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$

is an orthonormal and complete set on $(-L, L)$.

(b) $\left\{\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$

is an orthonormal and complete set on $(0, L)$.

(c) $\left\{\sqrt{\frac{1}{L}}, \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$

is an orthonormal and complete set on $(0, L)$.

The weight function $\omega(x) = 1$ in each case above.

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