PHYSNET

MISN-0-481

ORTHOGONAL CURVILINEAR COORDINATES

Math Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

ORTHOGONAL CURVILINEAR COORDINATES

by

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Title: **Orthogonal Curvilinear Coordinates**

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Input Skills:

1. Vocabulary: coordinate system, cartesian coordinate system, gradient, divergence, curl, laplacian.
2. Unknown: assume (MISN-0-480).

Output Skills (Knowledge):

K1. Define or explain the terms and concepts as follows: coordinate transformation, transformation equations, curvilinear coordinates, scale factor, orthogonal coordinates, differential elements of arc length in curvilinear coordinates, Jacobian of a coordinate transformation, spherical polar coordinates (r,θ,ϕ), differential element of volume in curvilinear coordinates.

K2. Write down from memory the transformation equations between rectangular coordinates (x,y,z) and each of the following: spherical polar coordinates (r,θ,ϕ) and circular cylindrical coordinates (ρ,ϕ,z).

Output Skills (Rule Application):

R1. Compute scale factors, unit vectors, arc lengths, surface and volume elements for each of the curvilinear coordinate systems in K2.

R2. Compute the gradient, divergence, curl, and laplacian in each of the curvilinear coordinate systems in K2.

External Resources (Required):

1. G. Arfken, Mathematical Methods for Physicist, Academic Press (1995).
2. Schaum's Outline: Murray Spiegel, *Theory and Problems of Advanced Mathematics for Scientists and Engineers*, McGraw-Hill Book Co. (1971).

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1. Introduction

Previously, most of your mathematical work in physics has centered around the use of rectangular coordinate systems. However, any constraint on or symmetry of a system can make mathematical analysis easier in another coordinate system. For instance, if we have a central force $\vec{F}(r) = F(r)\hat{r}$, such as gravitational or electrostatic force, rectangular coordinates may cause avoidable mathematical difficulties. I say "avoidable" because some of the mathematical difficulties can be avoided by using a coordinate system in which the radial distance is taken to be one of the coordinates, as in spherical polar coordinates. Later in the course, some interesting physical systems will be analyzed using several different curvilinear coordinate systems. Orthogonal curvilinear coordinate systems of various types turn out to be extremely useful in theoretical physics. In this unit, we treat spherical polar and circular cylindrical coordinate systems.

2. Procedures

1. Read the introduction to chapter 2 and sections 2.1 through 2.5 of Arfken.

2. Read pages 127-129 of Spiegel beginning with the section entitled "Orthogonal Curvilinear Coordinates. Jacobians."

3. Underline in the texts or write out the definitions and explanations of the terms and concepts of Output Skill K1.

4. Read through Solved Problems 5.40 to 5.43 of Spiegel on curvilinear coordinates and Jacobians of transformations.

5. Write down either set of formulas for gradient, divergence, curl, and Laplacian in curvilinear coordinates from Output Skill R2. These formulas can be consulted when solving problems and for the unit test.

6. Solve these problems in Arfken:

   2.1.2 (Scale factors and arc length for spherical polar coordinates)

   2.2.3 (Unit vectors for orthogonal curvilinear coordinates)[1]

   2.5.1 (Spherical polar unit vectors)

   2.5.23 (Curl in spherical polar coordinates)

   2.4.12 (Curl in circular cylindrical coordinates) Note: $\vec{B} = \vec{\nabla} \times \vec{A}$.

7. Solve these Supplementary Problem in Spiegel:

   5.92 ($\vec{\nabla}\phi$ and $\vec{\nabla} \times \vec{A}$ in spherical polar coordinates)

8. Use Eq. 2.10 in Arfken to write down the expressions for the surface elements of all three coordinate surfaces in each of the coordinate systems in Output Skill K2. Do the same thing using Fig. 5-25 in Spiegel (instead of 2.10 in Arfken). Compare the two sets of results. (Problem 2.1.2 part (b) in Arfken requires the use of Fig. 5-25 in Spiegel in order to evaluate $ds_i = h_i dq_i$).

Acknowledgments

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[1]Note: The result of problem 2.2.3 of Arfken is important. It shows that the orthogonal unit vectors in a curvilinear coordinate system can be written as:

$$\hat{a}_i = \frac{1}{h_i}\frac{\partial \vec{r}}{\partial q_i}\,.$$

This result should be used in problem 2.5.1 of Arfken.