PHYSNET

MISN-0-478

VECTOR ALGEBRA: A REVIEW

Math

Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

VECTOR ALGEBRA: A REVIEW

by

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Title: **Vector Algebra: a Review**

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Length: 2 hr; 11 pages

Input Skills:

1. Be able to use simple trigonometric functions.

Output Skills (Knowledge):

K1. Write the definition or explain each of the following terms or concepts in one or two sentences: vector, scalar, equality of vectors, vector addition and subtraction, null vector, unit vector, scalar product, vector function, rectangular unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$, components of vector, basis and span, direction cosines of coordinate transformation, orthogonality condition for direction cosines, a_{ij}, vector product.

K2. Write the definition of a vector field and scalar field and test whether a given vector or scalar satisfies the definition by considering of rotation of the coordinates.

Output Skills (Rule Application):

R1. Compute scalar and vector products given two vectors with numerical values for their components.

R2. Compute scalar and vector triple products given three vectors with numerical values for their components.

R3. Prove simple vector identities starting with the laws of vector algebra, scalar products, and vector products.

External Resources (Required):

1. G. Arfken, *Mathematical Methods for Physicists*, Academic Press (1995).
2. Schaum's Outline: Murray Spiegel, *Theory and Problems of Advanced Mathematics for Scientist and Engineers*, McGraw-Hill Book Co. (1971).

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1. Introduction

Most of the physical quantities studied in undergraduate physics are either scalar or vector fields. An example of a scalar field would be the distribution of temperatures over a weather map at a given time. An example is shown in Fig. 1.

Each point on the map has a certain temperature, but only a few examples are usually given. A vector field would be the wind speed and direction over a weather map at a given time. An example is shown in Fig. 2.

The arrows give the wind direction. The rules for algebraically manipulating combinations of scalars and vectors are also presented. Most of this unit should be review so you probably can finish it in three or four days.

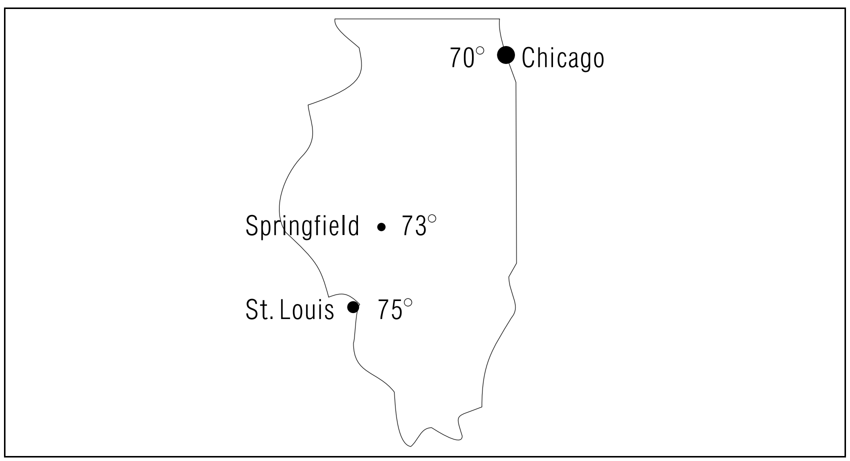


Figure 1. .

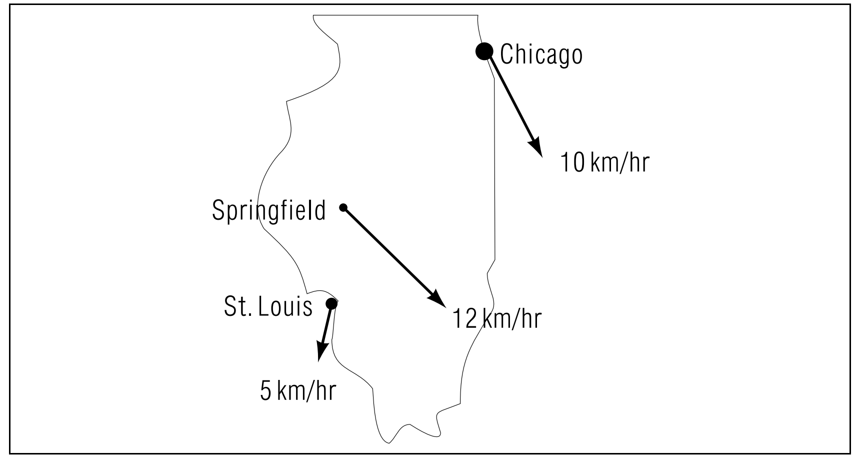


Figure 2. .

2. Procedures

1. Read pages 121 - 125 of Spiegel. Do not read the last section on page 125.

2. Read sections 1.1 through 1.5 of Arfken.

3. Underline in the texts or write out the definitions and explanations of the terms and concepts of Ouput Skill K1.

4. ▷ Write out the definitions of a vector field and scalar field as given in the Supplementary Notes.

5. Read through the Solved Problems 5.1 to 5.25 in Spiegel on vector algebra, scalar product, vector product, and triple product.

   Note: Be prepared to use either geometric techniques with vectors as directed line segments or algebraic techniques with vectors in component form.

6. ▷ Solve these problems in Arfken:

   1.1.7 (Vector addition)

   1.4.1 (Scalar and vector product)

   1.5.4 (Triple product)

7. ▷ Solve these Supplementary Problems in Spiegel:

5.50 (Unit vector)

5.63 (Triple product)

5.53 (Scalar product)

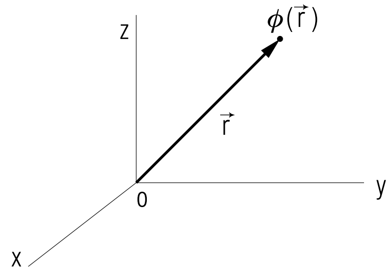
5.64 (Triple product)

5.59 (Cross product)

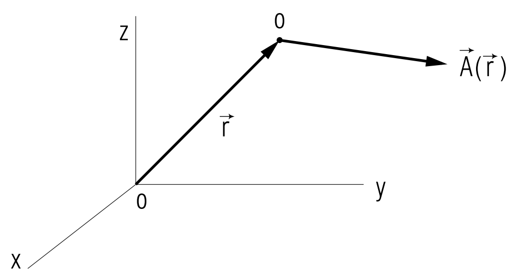
3. Supplementary Notes

Scalar and Vector Fields

Spiegel defines a *scalar field* as a scalar function $\phi(\vec{r}) = \phi(x, y, z)$ which associates a scalar with each point in a region. Here O is the origin of the (x, y, z) coordinate system. The quantity $\phi(\vec{r})$ is the value of a scalar physical quantity, like temperature, at the point $\vec{r}$.



Spiegel defines a vector field as a vector function $\vec{A}(\vec{r}) = A(x, y, z)$ which associates a vector, like force on a particle, at each point in a region.



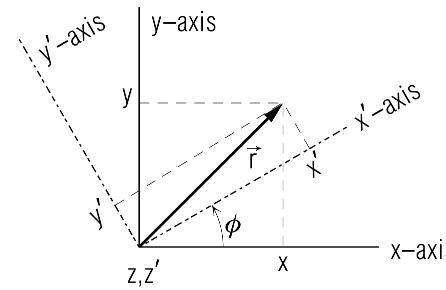
However, Arfken makes the definition more precise. His definition says that a scalar field must also be invariant under rotations of coordinates. In order to explore the meaning of this definition, redefine the coordinates as:

$$
\begin{aligned}
x &= x_1 \quad & x' &= x_1' \\
y &= x_2 \quad & y' &= x_2' \\
z &= x_3 \quad & z' &= x_3'
\end{aligned}
$$

Then, if the coordinate axes are rotated about the z-axis in a counterclockwise sense through an angle ϕ, the components $(x,\ y,\ z)$ of the position vector transform to $(x',\ y',\ z')$ as

$$
\begin{aligned}
x' &= x\cos\phi + \sin\phi \\
y' &= -x\sin\phi + y\cos\phi \\
z' &= z
\end{aligned} \tag{1}
$$

For simplicity, we shall restrict ourselves to rotations about the z-axis so the z-coordinate will not be included.



We define the direction cosines as in Arfken:

$$
\begin{aligned}
a_{11} &= a_{22} = \cos\phi \\
a_{12} &= -a_{21} = \sin\phi\,.
\end{aligned} \tag{2}
$$

Then, eq. (1) becomes

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 \\
x_2' &= a_{21}x_1 + a_{22}x_2\,.
\end{aligned} \tag{3}
$$

Then, $\phi(\vec{r})$ is a scalar field if it is invariant after the rotation,

$$\phi(x_1', x_2', x_3') = \phi(x_1, x_2 x_3) \tag{4}$$

However, a vector field is one which transforms like the position vector in eq. (1). Let:

$$\vec{A}(\vec{r}) = A_1(x_1, x_2, x_3)\,\hat{\imath} + A_2(x_1, x_2, x_3)\,\hat{\jmath} + A_3(x_1, x_2, x_3)\,\hat{k}\,, \tag{5}$$

in an obvious notation. Thus, $\vec{A}$ will be a vector field if:

$$A_1'(x_1', x_2', x_3') = a_{11}A_1(x_1, x_2, x_3) + a_{12}A_2(x_1, x_2, x_3)$$

$$A_2'(x_1', x_2', x_3') = a_{21}A_1(x_1, x_2, x_3) + a_{22}A_2(x_1, x_2, x_3)$$

or

$$
\begin{aligned}
A_1' &= a_{11}A_1 + a_{12}A_2 \\
A_2' &= a_{21}A_1 + a_{22}A_2\,.
\end{aligned} \tag{6}
$$

In this special case, $A'_3 = A_3$.

$\triangleright$ Given a pair of quantities $(-y, x)$, show that they are components of a two-dimensional vector, $\vec{V} = -y\hat{i} + x\hat{j}$, by demonstrating their form invariance under rotations

$\triangleright$ Given the pair of quantities $(-y, x)$, show that they are not components of a two-dimensional vector.

Acknowledgments

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