PHYSNET

MISN-0-473

# TENSOR ANALYSIS

# Relativity

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI



TENSOR ANALYSIS

by

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Title: **Tensor Analysis**

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**Input Skills**:

1. Unknown: assume (MISN-0-472).

**Output Skills (Knowledge)**:

K1. Define: (a) mixed components of a general tensor; (b) metric of a general spacetime; (c) Christoffel symbols (of the 2nd kind); (d) absolute derivative of a vector; (e) covariant derivative of a tensor; (f) Riemann curvature tensor; (g) Gaussian coordinates; (h) geodesic coordinates (i) Fermi coordinates.

K2. Write down the geodesic equation and the geodesic deviation equation.

K3. Given that $x^\mu$ are spacetime coordinates, $T^{\mu\cdots}_{\nu\cdots}$ is a general tensor, $g_{\mu\nu}$ is the metric, $ds$ is the invariant interval and $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols of the second kind, identify and prove which of the following are general tensors: (a) $x^\mu$; (b) $\Delta x^\mu$; (c) $dx^\mu$; (d) $g_{\mu\nu}$; (e) $\partial_\rho T^{\mu\cdots}_{\nu\cdots}$; (f) $dx^\mu/ds$; (g) $dT^{\mu\cdots}_{\nu\cdots}/ds$; (h) $\Gamma^\mu_{\nu\rho}$; (i) $DA^\mu/ds$, $A^\mu$ =vector (no proof) (j) $dT^{\mu\cdots}_{\nu\cdots;\rho}/ds$ (no proof)

**Output Skills (Problem Solving)**:

S1. Solve problems of the types given in the module's *Procedures* section.

**External Resources (Required)**:

1. W. Rindler, *Essential Relativity*, van Nostrand (1977).



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## 1. Introduction

Although a very qualitative understanding of General Relativity may be had without the use of tensors, an in-depth quantitative understanding makes this use mandatory. There are at least two reasons for this. First, the requirement that the laws of physics take the same form in all space - time coordinate systems is most easily satisfied by using tensor equations; and, second, the use of tensor notation greatly reduces the number and complexity of equations that need to be written down (just as the use of vector notation reduces the number of equations - usually from three to one - and makes them more highly compact). It is thus necessary to acquire at least a rudimentary knowledge of general tensor analysis. That is the purpose of this unit.

## 2. Procedures

1. Read section 8.1 of Rindler.

   Write down definitions of the quantities listed in Output Skill K1.

   Comments:

   1) Equation 8.13 of Rindler should read

   $$\Gamma^{\mu}_{\nu\sigma} = \Gamma^{\mu}_{\sigma\nu} = \frac{1}{2} g^{\mu\tau}(\partial_\sigma g_{\tau\nu} + \partial_\nu g_{\tau\sigma} - \partial_\tau g_{\nu\sigma})$$

   2) The covariant derivation of an arbitrary tensor is defined by:

   $$T^{\mu\dots}_{\nu\dots,\sigma} = \partial_\sigma T^{\mu\dots}_{\nu\dots} + (\Gamma^{\mu}_{\tau\sigma} T^{\tau\dots}_{\nu\dots} + \dots) - (\Gamma^{\lambda}_{\nu\sigma} T^{\mu\dots}_{\lambda\dots} + \dots)$$

   The first parenthesis contains one term for each contravariant index while the second parenthesis contains one term for each covariant index.

2. The geodesic equation and geodesic deviation equations are given by Rindler's equations 8.15 and 8.22 respectively.





(Optional) Read §13.4 of Misner, Thorne and Wheeler to see how the geodesic equation arises from extremizing the proper time along the world-line between two events.

▷ Exercise - the geodesic equation can be written in the form

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\nu\sigma} \frac{ds^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

where $\lambda$ is any real parameter which is continuous and monotonic along the curve.

Use this expression to show that

$$\frac{d}{d\lambda}\left(\frac{dS}{d\lambda}\right)^2 = 0$$

where

$$(dS)^2 = g_{\mu\nu} dx^\mu dx^\nu$$

This Exercise shows that $(dS/d\lambda)^2$ is a constant and hence the sign of $(dS)^2$ is constant along the curve. Consequently, geodesics divide naturally into three types:

1) timelike - along which $(dS)^2 > 0$
2) null - along which $(dS)^2 = 0$
3) spacelike - along which $(dS)^2 < 0$

3. ▷ Exercise – Determine the transformation laws for the quantities a - h in Output Skill K3. Some of these are discussed in section 8.1 of Rindler. The others you will have to determine on your own.

   The quantities in i and j are tensors, but the proofs are very tedious and not required for this unit.

   ▷ Exercise – As a special case of j show that the covariant derivative (or gradient) of a scalar is a first rank tensor (or vector).

   ▷ Exercise – Given that $V_\mu$ are the covariant components of a vector, show that the curl $\partial_\nu V_\mu - \partial_\mu V_\nu$ is a 2nd rank tensor although $\partial_\nu V_\mu$ and $\partial_\mu V_\nu$ are not separately tensors (see item e of Output Skill K3). Hint: Consider the quantity

   $$V_{\mu_S\lambda} - V_{\lambda_S\mu}$$

   ▷ Exercise – Show that the quantity

   $$\partial_\lambda A_{\mu\nu} + \partial_\nu A_{\lambda\mu} + \partial_\mu A_{\nu\lambda}$$

is a 3rd rank tensor if $A_{\mu\nu}$ is a 2nd rank antisymmetric tensor. Hint: Consider the sum

$$A_{\mu\nu S\lambda} + A_{\lambda\mu S\nu} + A_{\nu\lambda S\mu}$$

4. ▷ Exercise - Prove the following properties of covariant derivatives

1) The covariant derivative of a linear combination of tensors (with constant coefficients) is the same linear combination of the covariant derivatives.
2) The covariant derivative of a product of tensors obeys Leibniz rule. Note: Leibniz rule for functions is

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

3) The covariant derivative of a contracted tensor is the contraction of the covariant derivative.

▷ Exercise – Let $\xi^\alpha$ be pseudo-Euclidian coordinates in a LIF so that the equation of a straight line is given by

$$\frac{d^2\xi^\alpha}{dS^2} = 0$$

and the proper time between two nearby events is given by

$$(dS)^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$$

where $\eta_{\alpha\beta}$ is the Minkowski metric $(1, -1, -1, -1)$. By transforming to arbitrary coordinates, show that in the new coordinates, the metric is given by

$$g_{\mu\nu} = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

and the Christoffel symbols of the second kind are given by

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}$$

▷ Exercise – Show that along any world-line $x^\mu = x^\mu(\lambda)$ geodesic or not, where $S$ is the proper time along the world-line, that

$$g_{\mu\nu}\frac{dx^\mu}{dS}\frac{D^2x^\mu}{dS^2} = 0$$





Hint: Start with the definition of the metric

$$(dS)^2 = g_{\mu\nu} dx^\mu dx^\nu$$

which along any world-line can be written

$$g_{\mu\nu}\frac{dx^\mu}{dS}\frac{dx^\nu}{dS} = 1$$

Now differentiate one time with respect to $S$ and rearrange.

Comment: The last Exercise shows that the (general) 4 - velocity and (general) 4 - acceleration of any particle are always space-time orthogonal.

▷ Work Exercises 8.1 and 8.3 on p. 269 of Rindler.

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