PHYSNET

MISN-0-471

RELATIVISTIC MECHANICS

Relativity

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

RELATIVISTIC MECHANICS

by

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Title: **Relativistic Mechanics**

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Input Skills:

1. Vocabulary: 4-vector, 4-velocity, 4-acceleration, covariance (MISN-0-469).
2. State the transformation property of 3-forces (MISN-0-470).

Output Skills (Knowledge):

K1. Give a plausibility argument (including a definition of 4-momentum) for the relativistic form of Newton's 2nd law. Base the argument on the classical form of the 2nd law, the special relativity postulate, and the 3-force transformation law.

K2. (a) Derive the form of the Minkowski 4-force using plausibility arguments. (b) Determine the transformation equations for the 3-force. (c) Express the relativistic form of Newton's 2nd law in terms of longitudinal and transverse masses for the appropriate special cases.

K3. (a) Given an interpretation of the time component of the momentum 4-vector in terms of the particle's energy. (b) Establish conservation of 4-momentum for a closed system.

Output Skills (Problem Solving):

S1. Given information about the particles involved in a collision or decay use conservation of 4-momentum and the center of mass frame to determine quantities (momenta, energies, velocities) related to the process.

External Resources (Required):

1. W. Rindler, *Essential Relativity*, Van Nostrand (1977).

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1. Introduction

This is the final unit on the special relativity part of this course. It is concerned with the modification of Newton's 2nd law required to bring it into conformity with the special relativity postulate, with conservation of energy and momentum and with applications of these principles to various mechanics problems. It should be kept in mind that conservation of energy and momentum is, in fact, more fundamental than is the 2nd law since it applies to all closed physical systems whether made up of particles or not.

2. Procedures

1. Read Rindler, section 5.1.

Newton's 2nd law of motion can be written in several equivalent ways.

$$\vec{f} = m\vec{a} = m\frac{d\vec{u}}{dt} = \frac{d\vec{p}}{dt},\ \vec{p} = m\vec{u}$$

where $\vec{f}$ is the force acting on the (constant) mass m and resulting in the acceleration $\vec{a}$. It is desired to find the proper relativistic generalization of this law. There are many possible candidates and the correct choice must ultimately be determined by its consistency with observations. However, there are several constraints upon the form of the relativistic expression that drastically limits the possibilities.

1) The relativistic expression should reduce to the classical expression in the now relativistic limit ($u \ll 1$).
2) The relativistic expression should obey the special relativity postulate (i.e. have the same form in all inertial frames).
3) The 3-force ($\vec{f}$) transforms like $1/\gamma_u$ times the spatial part of a 4-vector.

An obvious way to obtain a 4-vector (as required by constraint 3) analogous to the 3-momentum is to define the 4-momentum by

$$P^\mu = m_0 U^\mu$$

where m is a scalar property of the particle - its rest mass. The components of the 4-momentum can be written in the form

$$P^\mu = m_0\gamma_u(1,\vec{u}) \equiv (P^\circ, \vec{P})$$

Thus the spatial part of the 4-momentum is

$$\vec{P} = m_0\gamma_u\vec{u}$$

which in the non-relativistic limit reduces to the classical 3-momentum if we identify m with the Newtonian inertial mass

$$\vec{P} \Rightarrow m_0\vec{u} = \vec{p}$$

$$m_0 = m \text{ (Newton)}$$

Thus the spatial part of the 4-momentum seems to be a likely candidate to use on the right hand side of the relativistic equation of motion. In fact it is tempting to write down

$$\vec{f} = \frac{d\vec{P}}{dt}$$

This clearly has the correct non-relativistic limit (constraint 1). However, it is not so clear that it satisfies constraint 2. To see that it does consider the proper time τ of the particle

$$(d\tau)^2 = (d\tau)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

where (dt, dx, dy, dz) is a space-time displacement along the world line of the particle. $(d\tau)^2$ is a Lorentz scalar by construction and can be written in the form

$$(d\tau)^2 = (dt)^2\left[1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2\right]$$

$$= (dt)^2(1-u^2) = \frac{(dt)^2}{\gamma_u^2}$$

Thus

$$\frac{d\vec{P}}{dt} = (1/\gamma_u)\frac{d\vec{P}}{d\tau}$$

$$= (1/\gamma_u) \text{ times the spatial part of a 4-vector}$$

This observation along with the known transformation character of $\vec{f}$ establishes that the expression

$$\vec{f} = \frac{d\vec{P}}{dt}, \ \vec{P} = m_0 \gamma_u \vec{u}$$

satisfies the special relativity postulate (constraint 2). This is the desired relativistic equation of motion. Its correctness has indeed been substantiated by numerous comparisons with observations.

Notes:

1. If one uses for $\vec{f}$ an expression (such as Newton's law of gravitation) which comes from a theory which is not Lorentz covariant then, of course, the relativistic equation of motion will not be invariant under Lorentz transformations.
2. In the instantaneous rest frame of the particle $\vec{u} = 0$ and hence $d\tau = dt$.

   Thus $d\tau$ is the time interval recorded by a clock traveling with the particle.
3. It is common practice to identify

   $$m = m_0 \gamma_u$$

   as the relativistic inertial mass and to write the relativistic momentum in the form

   $$\vec{P} = m\vec{u}$$

   Although this maintains the symbolism of classical mechanics it is of dubious value. It is undoubtedly simpler and more to the point to refer to the rest mass of the particle as *the* mass of the particle (as is universally done in high energy physics) and to write the relativistic momentum as

   $$\vec{P} = m_0 \gamma_u \vec{u}$$

▷ Exercise - Show that

$$P^\mu P_\mu = (P^\circ)^2 - \vec{P}^2 = m_0^2$$

2. a. Although the relativistic equation of motion

$$\vec{f} = \frac{d\vec{P}}{dt}$$

is covariant (both sides have the same transformation rule) it is not manifestly so. In fact, it would not be possible to transform this equation alone from one inertial frame to another since the Lorentz transformation requires the time - component of a 4-vector as well as the spatial part. It is thus desired to obtain a manifestly covariant form of the equation of motion. This is easily done by recalling that

$$\gamma_u \vec{f} = \gamma_u \frac{d\vec{P}}{dt} = \frac{d\vec{P}}{d\tau}$$

Hence the manifestly covariant expression must be of the form

$$F^\mu = \frac{dP^\mu}{d\tau}$$

with

$$F^\mu = (F^o, \gamma_u \vec{f}) = \text{ Minkowski 4-force}$$

The meaning of F^o can be determined as follows.

$$F^o = \frac{dP^o}{d\tau} = \gamma_u \frac{dP^o}{dt}$$

Now since

$$(P^o)^2 = m_0^2 + \vec{P}^2$$

differentiating gives

$$2P^o \frac{dP^o}{dt} = 2\vec{P} \cdot \frac{d\vec{P}}{dt}$$

or

$$\frac{dP^o}{dt} = \frac{\vec{P}}{P^o} \cdot \frac{d\vec{P}}{dt} = \frac{m_0 \gamma_u \vec{u}}{m_0 \gamma_u} \cdot \vec{f} = \vec{u} \cdot \vec{f}$$

Hence

$$F^o = \gamma_u \vec{u} \cdot \vec{f}$$

so that

$$F^\mu = \gamma_u (\vec{u} \cdot \vec{f}, \vec{f})$$

Note: The time component of the Minkowski force is just γ_u times the power expended by the force $\vec{f}$. Thus force and power (except for a factor of γ_u) are just different components of the same 4-vector. In cases where the rest mass m_0 is not constant (because of thermal heating, for example) the time component of the 4-force is γ_u times the total rate at which energy is supplied to the particle.

b. ▷ Exercise - Use the Minkowski force to show that for a boost at speed v in the $+x$-direction the 3-force transformation equations are

$$f'_x = \frac{f_x - v(\vec{u} \cdot \vec{f})}{1 - vu_x}, \; f'_y = \frac{f_y}{\gamma(1 - vu_x)}, \text{ etc.}$$

Hint: See the Exercise on p. 9 of MISN-0-469.

▷ Work problem 5 - 8 in Rindler,

c. ▷ Exercise - Differentiate the relativistic momentum

$$\vec{P} = m_0 \gamma_u \vec{u}$$

to show that the relativistic equation of motion can be written in the form

$$\vec{f} = m_0 \gamma_u \vec{a} + m_0 \gamma_u^3 (\vec{u} \cdot \vec{a}) \vec{u}$$

Note: The force $\vec{f}$ and the acceleration $\vec{a}$ are in general not parallel as they are in Newtonian Mechanics.

▷ Exercise - Use the result of the previous exercise to show that

a) $\vec{u} \cdot \vec{f} = m_0 \gamma_u^3 \vec{u} \cdot \vec{a}$
Hint: See the exercise on p. 7 of MISN-0-466.

b) $\vec{f} = m_0 \gamma_u \vec{a} + (\vec{u} \cdot \vec{f}) \vec{u}$

▷ Exercise - Show that

a) $\vec{f} = m_0 \gamma_u \vec{a}$ if $\vec{f}$ is perpendicular to $\vec{u}$

b) $\vec{f} = m_0 \gamma_u \vec{a}$ if $\vec{f}$ is parallel to $\vec{u}$

Note: These are the only two cases when $\vec{f}$ and $\vec{a}$ are parallel. Historically $m_0 \gamma_u$ was called the transverse mass and $m_0 \gamma_u^3$ the longitudinal mass.

3. a. It was shown earlier that the power expended by a force $\vec{f}$ is equal to the time derivative of the zeroth component of the 4-momentum

$$\vec{u} \cdot \vec{f} = \frac{dP^o}{dt}$$

One might suspect then that P^o is the relativistic generalization of the kinetic energy of the particle. However, expansion of P^o in powers of u yields

$$P^o = m_0 \gamma_u = m_0 (1 - u^2)^{-1/2}$$

$$= m_0 + 1/2 m_0 u^2 + 3/8 m_0 u^4 + \ldots$$

Hence the non-relativistic limit of P^o differs from the classical kinetic energy by the rest mass of the particle m_0. The relativistic kinetic energy is thus defined by

$$T = P^o - m_0 = 1/2 m_0 u^2 + 3/8 m_0 u^4 + \ldots$$

For a single stable particle m_0 is a constant and could be disregarded since additive constants are not important in energy considerations. However, for a system of particles the individual rest masses may change so that it is necessary to interpret P^o as the total energy.

$$P^o = E = m_0 + T$$

The total energy for a particle thus consists of two parts - the rest mass energy and the kinetic energy.

▷ Exercise - Assume T, P , m_0 and u have units of energy, momentum, mass and velocity respectively. Insert c's into the equation

$$E = m_0 + T = m_0 + 1/2 m_0 u^2 + 3/8 m_0 u^4 + \ldots$$

so that

a) all terms have units of momentum

b) all terms have units of energy.

Note: The 4-momentum can be written in several equivalent forms

$$P^\nu = m_0 U^\mu = m_0 \gamma_u (1, \vec{u}) = (P^o, \vec{P}) = (E, \vec{P})$$

From these expressions several useful relations (which have been used previously) can be derived

1) $E = m_0 \gamma_u = m, \; \vec{P} = m_0 \gamma_u \vec{u} = m \vec{u}$

2) $\vec{P}/E = \vec{u}$

3) $P^\mu P_\mu = E^2 - \vec{P}^2 = m_0^2$ or $E^2 = m_0^2 + \vec{P}^2$

▷ Exercise - Determine the speeds of the projectiles in each of the following accelerators. (Give the value of $1 - v$. See p. 7 of MISN-0-466).

a. Stanford Linear Accelerator (Stanford, Calif.)
electrons at 20 GeV

b. Serpukhov (USSR)
protons at 70 GeV

c. National Accelerator Laboratory (Batavia, Ill.)
protons at 400 GeV

Note: $1\,\text{GeV} = 10^9\,\text{eV}$, $1\,\text{MeV} = 10^6\,\text{eV}$

$m_e = 0.51\,\text{MeV}$

$m_b p = 940\,\text{MeV}$

b. Conservation of energy and momentum can be approached in two ways. The best way is to consider the homogeneity of space-time. This homogeneity requires any theory to be invariant under space-time translations which in turn leads to conservation of energy and momentum for all closed systems. Unfortunately the details of this analysis are beyond the scope of this course.

The alternate procedure is to apply Newton's 2nd law to a closed system of particles and show that energy and momentum must be conserved. This procedure is less satisfying for a number of reasons but it will be followed here. For a single particle the relativistic form of Newton's 2nd law is

$$\vec{f} = \frac{d\vec{P}}{dt}$$

For a system of particles this expression can be summed over all particles so that

$$\sum_i \vec{f}_i = \frac{d}{dt}\sum_i \vec{P}_i$$

Now for a closed system the total force on the system is zero so that

$$\frac{d}{dt}\sum_i \vec{P}_i = 0$$

which implies that the total momentum is a constant (i.e. momentum is conserved).

Note: One glaring deficiency of this procedure is that it assumes that the number and character of the particles does not change. On the other hand it is known that the momentum of a closed system is conserved even if the number of particles does change by way of creation and/or annihilation processes.

The conservation of the spatial part of the 4-momentum in all inertial systems implies the conservation of the temporal part as well. Read the material in square brackets at the top of page 80 in Rindler. Thus 4-momentum is conserved for a closed system.

Rindler, as well as many other authors, follows a different approach to relativistic dynamics. They begin assuming conservation of momentum and proceed to establish its consequences. There are some very definite advantages to that procedure. However, I have chosen to follow a path that more closely parallels the typical introductory development of classical mechanics.

(Optional) Read Rindler, sections 5.3 - 5.6.

4. Read Rindler, section 5.7. The center of mass frame is sometimes called the zero momentum frame. Although the latter name is undoubtedly better the use of center of mass frame is so embedded in the literature of physics that it is futile to try to avoid its usage.

Read Rindler, sections 5.8 and 5.9

▷ Exercise - Consider the production process wherein a particle of mass m_1 collides with a target particle of mass m_2 (at rest in the lab) to produce a collection of particles of masses m_3, m_4, ...

$$1 + 2 \Rightarrow 3 + 4 + \ldots$$

Use conservation of 4-momentum and the invariance of its square ($P^\mu P_\mu$) to show that the threshold energy (in the lab frame) for this process is

$$E = \frac{(m_3 + m_4 + \ldots)^2 - (m_1^2 + m_2^2)}{2m_2}$$

▷ Exercise - Use the result of the preceding exercise to show that the threshold energy for pair production by electron - electron collision

$$e^- + e^- \Rightarrow e^- + e^- + e^- + e^+$$

is seven times the rest mass energy of an electron. Note: the rest mass of a positron (e^+) is the same as that of an electron (e^-).

▷ Exercise - A moving kaon (K meson) decays into two pions (π meson)

$$K^+ \Rightarrow \pi^+ + \pi^0$$

If one of the pions is left at rest what was the energy of the kaon and what is the energy of the other pion?

$$m_K = 494\,\text{MeV}$$

$$m_\pi = 137\,\text{MeV}$$

Read Rindler, sections 5.11 and 5.l2. Note that the 4-momentum of a photon satisfies these relations

$$P^\mu = (E, \vec{P}) = h\nu(1, \hat{n})$$

and

$$P^\mu P_\mu = E^2 - \vec{P}^2 = 0$$

where h = Planck's constant
ν = frequency = $1/\lambda$ λ = wavelength
$\hat{n}$ = unit vector in the propagation direction

▷ Exercise - A photon rocket is to be accelerated from rest to a speed such that $\gamma_u = 10$ by converting part of the mass of the rocket into radiation (photons) which is directed out the rocket's exhaust. What fraction f of the mass must be so converted? (Assume 100% efficiency.)

Ans. $f = 1 + \gamma_u - \sqrt{\gamma_u^2 + 1} \approx 0.95$

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