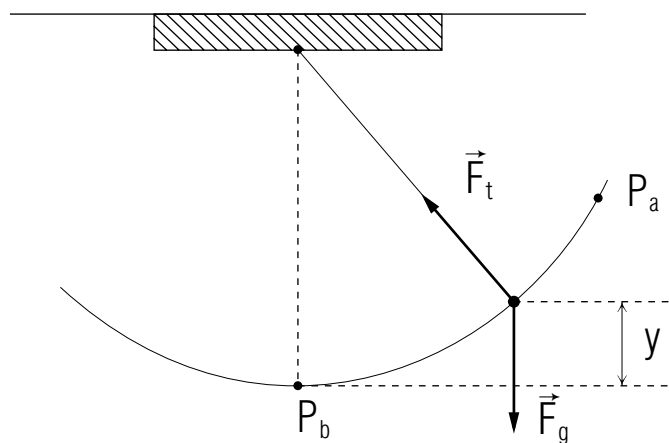


POTENTIAL ENERGY AND CONSERVATION OF ENERGY



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

POTENTIAL ENERGY AND CONSERVATION OF ENERGY

by

F. Reif, G. Brackett and J. Larkin

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- B. Conservation of Energy
- C. Work Done by Central Forces
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Title: **Potential Energy and Conservation Of Energy**

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Input Skills:

1. Vocabulary: gravitational force law (MISN-0-410); kinetic energy, work, integral (special summation) symbol \int (MISN-0-414).
2. State the work-kinetic energy relation (MISN-0-414).

Output Skills (Knowledge):

- K1. Vocabulary: potential energy, energy.
- K2. State the relationship between work and potential energy.
- K3. State the principle of conservation of energy.
- K4. Write expressions for gravitational potential energy and Coulomb potential energy.

Output Skills (Problem Solving):

- S1. Solve problems using these relations for a single particle:
- S2. Qualitatively or quantitatively relate a particle's position and its potential energy due to: (a) the gravitational force near the earth; (b) a gravitational or electric force due to a particle.
- S3. Apply conservation of energy to relate, quantitatively or qualitatively, a particle's position and speed.

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SECT.

A

 POTENTIAL ENERGY

DEPENDENCE OF WORK ON PROCESS

A particle can move from a point P_a to a point P_b by a variety of processes, i.e., it can move along many different paths and with many different speeds. How then does the work done on a particle by some force \vec{F} depend on the process by which the particle moves between the two points?

Consider first the simple case where a force \vec{F} on a particle is constant throughout a certain region of space, irrespective of the position or velocity of the particle. (For example, the force might be the gravitational force on a particle in a limited region near the surface of the earth.) By Relation (C-1) of Unit 414, the work done on the particle by this force \vec{F} along some path is then simply $F D_F$, where D_F is the numerical component along \vec{F} of the displacement \vec{D} of the particle from its initial to its final position. But the constant force \vec{F} has the same value throughout the entire region and the displacement \vec{D} depends only on the end points of the particle's path. Hence the work done on the particle by the force \vec{F} depends only on the end points of the particle's path, irrespective of the specific path traversed by the particle or the speed with which it moves. (For example, in Fig. A-1, the work done by the constant force \vec{F} between the points P_a and P_b is the same along any one of the indicated paths connecting these points.) Thus we arrive at this conclusion: If the force on a particle is constant throughout a region of space, the work done on the particle moving between two points is *independent* of the process (i.e., independent of the path and speed) by which the particle moves between these points.

As we shall see later, the work done on a particle by any *fundamental* force is also independent of the process by which the particle moves between two points. On the other hand, the work done on a particle may depend on the process when the particle is acted on by some empirical force, such as a frictional force. For example, the frictional force on a particle sliding along a surface has ordinarily a constant *magnitude*, but a direction always opposite to the successive small displacements of the particle along its path. Hence the (negative) work done by this frictional force has a larger magnitude if the particle travels along a longer path.

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Abstract:

Many forces, including all fundamental forces, depend in a simple way on the positions of the interacting particles. Hence the work done on a particle by such forces has quite simple properties. As a result, the particle can be characterized by a quantity, called its "energy," which remains constant in time (i.e., which is conserved). This principle of conservation of energy, discussed in this unit and elaborated further in the next, is one of the most important principles in all the natural sciences.

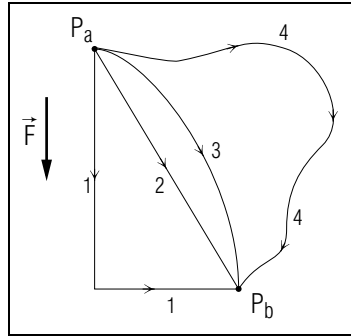


Fig. A-1: Work done on a particle along various paths between two points. (The arrow indicates the direction of a constant force \vec{F} .)

For instance, in Fig. A-1 the magnitude of the work done by the *frictional* force is larger along the longer path 4 than along the shorter path 2.

DEFINITION OF POTENTIAL ENERGY

Suppose that the force \vec{F} on a particle due to its interaction with certain other fixed particles is such that the work done on the particle by \vec{F} between two points is *independent of the process*. Then this work depends only on the end points P_a and P_b of the particle's path. As we shall show, this work W_{ab} can then be found very simply by comparing it with the work done by \vec{F} when the particle moves from each of these points to some standard state s where it is at rest at some standard position P_s .

Indeed, the particle can move from P_a to the standard position P_s along various paths. For example, if the particle moves from P_a to P_s along some fairly direct path (such as the path 1 in Fig. A-2), some work W_{as} is done on the particle by the force \vec{F} . Alternatively, if the particle moves from P_a to P_s by passing through the point P_b (as indicated by the path 2 in Fig. A-2), the work done on the particle by \vec{F} is $(W_{ab} + W_{bs})$,

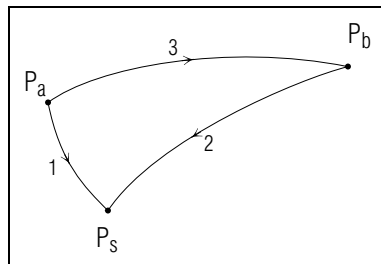


Fig. A-2: Path between two points and paths from these points to a standard position P_s .

the work done while moving from P_a to P_b plus the work done while moving from P_b to P_s . But since the work done in moving from P_a to P_s is independent of the process, the works done along these two different paths must be the same so that

$$W_{as} = W_{ab} + W_{bs}$$

or equivalently,

$$W_{ab} = W_{as} - W_{bs} \tag{A-1}$$

Suppose then that we have found the works W_{as}, W_{bs}, \dots done by \vec{F} when the particle moves from various points P_a, P_b, \dots to the chosen standard position P_s of the particle. Then we can immediately find the work done by \vec{F} between any two points, such as P_a and P_b , by the simple subtraction indicated in Eq. (A-1). To express this result in convenient form, let us introduce the abbreviation

$$\boxed{U_a = W_{as}.} \tag{A-2}$$

The quantity U is called the "potential energy" of the particle and its definition, Def. (A-2), can be stated in words:

Def.	<p>Potential energy: The potential energy U_a of a particle at some point P_a, due to its interaction with other particles, is the work W_{as} done on the particle by the force due to this interaction when the particle moves from P_a to some specified standard position P_s (while all other particles remain fixed).</p>	(A-3)
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With this definition, we can write the result Eq. (A-2) in the simple form

$$\boxed{W_{ab} = U_a - U_b} \tag{A-4}$$

Thus we arrive at this conclusion:

<p>If the work done on a particle is independent of the process, the work can be found from a difference of potential energies by the relation $W_{ab} = U_a - U_b$.</p>

(A-5)

DISCUSSION

By its definition $U_a = W_{as}$, the potential energy of a particle is, like any work, a number which can be expressed in terms of the unit joule and which can be positive, negative, or zero. For a specified choice of the standard position P_s , the potential energy U_a depends only on the position P_a of the particle (since the work W_{as} is independent of the process by which the particle moves from P_a to P_s). Furthermore,

$$\boxed{U_s = 0} \quad (\text{A-6})$$

since $W_{ss} = 0$ (i.e., since no work is done if the particle merely remains at the standard position). Thus the potential energy of a particle at the standard position is equal to zero.

The relation (A-4) states that the work done on a particle has a magnitude equal to that of the change in the potential energy of the particle, being positive if the potential energy decreases (so that the initial potential energy U_a is larger than the final potential energy U_b) and being negative if the potential energy increases. *

* We could also write Def. (A-3) as $W_{ab} = -(U_b - U_a) = \Delta U$, a result which shows that the work done on the particle is always opposite to the change ΔU of the potential energy of the particle.

Thus Eq. (A-4) shows explicitly that the work W_{ab} depends only on the initial and final positions of the particle, and can be found simply by subtracting the potential energies of the particle at these positions.

It is important to remember that the value of the potential energy depends on what point is chosen as the standard position. (On the other hand, the *difference* $U_a - U_b$ between the potential energies at two points does *not* depend on the choice of standard position, since it is always equal to the work W_{ab} done between these points.)

The relations (A-4) and (A-6) summarize all the properties of potential energy of the potential energy. These relations also imply the definition of the potential energy, Def. (A-2). [Indeed, if P_b denotes the standard position P_s , Eq. (A-4) implies that $W_{as} = U_a - U_s = U_a$ since $U_s = 0$ by Eq. (A-6).]

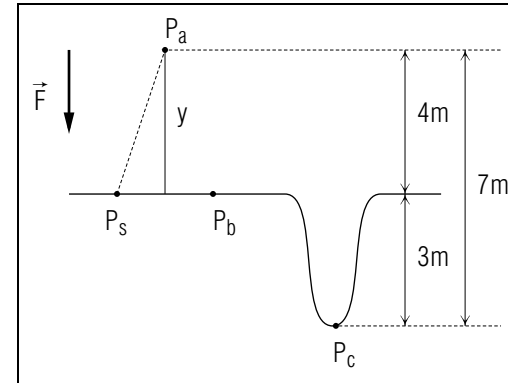


Fig. A-3: Gravitational potential energy and work near the surface of the earth.

POTENTIAL ENERGY DUE TO GRAVITY NEAR THE EARTH

To illustrate the preceding remarks, consider a particle of mass m located near the surface of the earth. To find the gravitational potential energy of the particle due to its interaction with the earth, we must first choose some convenient standard position P_s (e.g., some point at the surface of the earth). We can then find the potential energy of the particle located at the point P_a by using the definition $U_a = W_{as}$ of Eq. (A-2). Thus we need merely find the work W_{as} done on the particle by the gravitational force $\vec{F} = m\vec{g}$ when the particle moves from P_a to P_s . But since \vec{F} is constant near the surface of the earth, $U_a = W_{as} = F D_F = mg D_F$, where D_F is the numerical component along the downward gravitational force of the particle's displacement from P_a to P_s . As is apparent from Fig. A-3, D_F is equal to the vertical height y_a of the point P_a above the standard point P_s , this height being considered positive if the point P_a is above P_s (so that D_F is positive) and being considered negative if P_a is below P_s (so that D_F is negative). Thus we find that $U_a = mgy_a$. In other words, the gravitational potential energy U of a particle of mass m located at any height y above the standard position is simply

$$\boxed{U = mgy} \quad (\text{A-7})$$

Note that this gravitational potential energy increases with increasing height of the particle above the standard position.

Example A-1: Gravitational potential energy and work

Consider an apple of mass $m = 0.2 \text{ kg}$. Its weight is then $mg = (0.2 \text{ kg})(10 \text{ m/s}^2) = 2 \text{ N}$. Suppose that the apple is in a tree at a point

P_a 4 meter above a standard position P_s chosen at the surface of the ground. (See Fig. A-3). Then the height $y_a = 4$ meter and the gravitational potential energy of the apple is $U_a = (mg)y_a = (2\text{ N})(4\text{ m}) = 8\text{ joule}$. If the apple is lying at the point P_b at the foot of the tree at the same height as the standard position P_s , the height $y_b = 0$ and the gravitational potential energy of the apple is $U_b = 0$. If the apple is at the point P_c at the bottom of a trench 3 meter below the point P_s , the height $y_c = -3$ meter and the gravitational potential energy of the apple is $U_c = (2\text{ N})(-3\text{ m}) = -6\text{ joule}$.

If the apple moves from the point P_a to the point P_c , the work done on the apple by the gravitational force is then simply

$$W_{ac} = U_a - U_c = 8\text{ joule} - (-6\text{ joule}) = 14\text{ joule}$$

This result agrees with that obtained if we calculate directly the work $W_{ac} = FD_F = mgD_F$ by using $mg = 2$ and $D_F = 7$ meter for the numerical component of the downward displacement of the apple from the point P_a to the point P_c located 7 meter below the level of P_a .

Knowing About the Definition of Potential Energy

A-1 The charged particles in a transistor-radio battery exert on an electron an electric force \vec{F}_e which does work independent of process. Thus we can find the electron's potential energy U due to this force. (a) Suppose an electron moves through the radio from one battery terminal A to the other battery terminal B . In this process, the electron goes from a state a (where it is located at terminal A) to a state b (where it is located at terminal B). During this process, the work done on the electron by \vec{F}_e is $W_{ab} = -6.4 \times 10^{-18}\text{ J}$ (where J is the SI abbreviation for joule). If we choose the terminal B as the standard position, what is the electron's potential energy U_b at this position? What is its potential energy U_a at terminal A ? (b) Suppose instead that the electron moves from terminal A to terminal B along a straight wire. Is this process the same as that described in part (a)? Is the work W_{ab} done by \vec{F}_e in this process (and thus the potential energy U_a) the same? (Answer: 106) (Suggestion: [s-2])

Understanding ($W_{ab} = U_a - U_b$) (Cap. 1a)

A-2 *Statement and example:* Figure A-4 shows several positions of a baseball during a ball game, and gives for each position the ball's

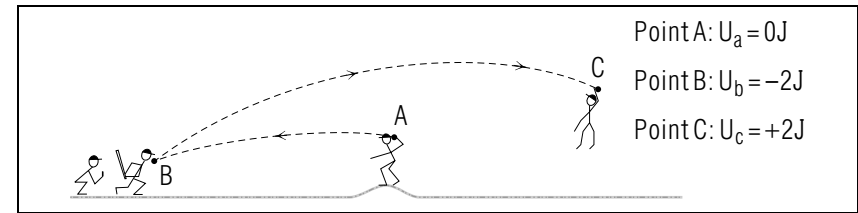


Fig. A-4.

gravitational potential energy due to the gravitational force \vec{F}_g . (The point A has been chosen as the standard position.) (a) Consider the works W_{ab} and W_{bc} done by \vec{F}_g on the ball as it moves from the point A to the point B , and from the point B to the point C . Express each work in terms of symbols for the ball's potential energy, and then find the value of this work. (b) In the next pitch, the ball travels from the standard position at the point A to a point D in the catcher's mitt. If the work done by \vec{F}_g during this motion is 3 J, what is the ball's gravitational potential energy U_d at the point D ? (Answer: 103)

A-3 *Meaning of U :* An efficiency expert, wishing to find the work done by the frictional force \vec{F}_f on a cart pushed between points on a factory floor, decides to find for each point the cart's potential energy due to \vec{F}_f . Briefly explain why this *cannot* be done. (Answer: 108)

A-4 *Comparing potential energy and work:* Consider again the ball described in problem A-2. If we choose the point C instead of the point A as the standard position, which of the following quantities has a different value? (a) The work W_{bc} done by \vec{F}_g as the ball moves from the point B to the point C . (b) The ball's gravitational potential energies U_b and U_c at these points. (c) The difference $U_b - U_c$ between these potential energies. (Answer: 101) (Suggestion: [s-10])

A-5 *Finding potential energy from work:* The charged deflection plates of an oscilloscope tube exert on an electron a constant electric force $\vec{F}_e = 1 \times 10^{-14}\text{ N}$ upward. The work done on the electron by this force is independent of process. (a) Find the works W_{xy} , W_{xz} , and W_{yz} done by \vec{F}_e on the electron as it moves between the points X , Y , and Z shown in Fig. A-5. (b) If we choose the point X as the standard position, what is the electron's potential energy due to \vec{F}_e at each point? (c) What is its potential energy at each point if we choose the point Y as the standard position? (Answer: 104) (Suggestion: [s-12])

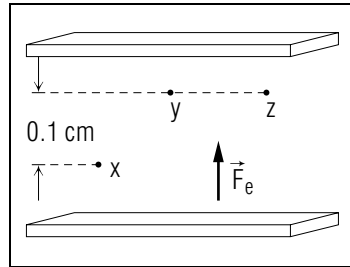


Fig. A-5.

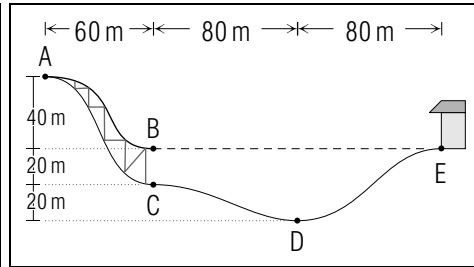


Fig. A-6.

largest? Where is it smallest? (*Answer: 105*)

A-6 *Dependence of potential energy on sign of work:* In each of the following cases the work done is independent of process, meaning independent of the path over which the work is done. For each case give the sign of the work and state whether the particle's final potential energy due to the force is larger than, equal to, or smaller than its initial potential energy. (a) The work done by the sun's gravitational force on a comet moving directly toward the sun. (b) The work done by the same force on a planet moving along a circular orbit around the sun. (c) The work done by a positively-charged nucleus exerting a repulsive Coulomb electric force on a positively-charged proton moving directly toward that nucleus.¹ (d) The work done by the earth's gravitational force on an elevator moving upward. (*Answer: 110*) (*Suggestion: [s-6]*)

Relating Potential Energy and Position (Cap. 2a)

A-7 What is the gravitational potential energy of a 50 kg skier at each of the points in Fig. A-6? Choose either the point *B*, *C*, or *E* as the standard position. (*Answer: 102*)

A-8 (a) A woman's gravitational potential energy is larger when she has coffee in the cafeteria than when she works in her office. Is the woman's office on a lower or higher floor than the cafeteria? (b) A car drives from a point at sea level on a Los Angeles beach to the top of a mountain pass and then down again to a point at sea level in Death Valley. During this trip, where is the car's gravitational potential energy

¹The Coulomb electric force between two charges has exactly the same mathematical form as the gravitational force between two masses. However, whereas the gravitational force is always attractive, the Coulomb force is repulsive if the two charges have the same sign, attractive if they are different. Note that no understanding of the force is needed for the problem at hand.

SECT.

B

 CONSERVATION OF ENERGY

As we know from Relation (B-4) of Unit 414, the change of the kinetic energy K of a particle is always related to the work W done by *all* forces acting on the particle so that

$$K_b - K_a = W_{ab} \quad (\text{B-1})$$

as the particle goes from any initial state a (specified by some initial position and velocity) to any final state b (specified by some final position and velocity). Suppose now that the work done on the particle by every force which does work is *independent* of the process. Then we can use Eq. (A-4) to express the work W_{ab} done on the particle in terms of the change of its potential energy by writing

$$W_{ab} = U_a - U_b \quad (\text{B-2})$$

where U is the potential energy of the particle due to *all* forces which do any work. By combining the relations (B-1) and (B-2), we then obtain

$$K_b - K_a = U_a - U_b \quad (\text{B-3})$$

In other words, any increase of the particle's kinetic energy is always equal to the corresponding decrease of its potential energy (and vice versa). Thus the *sum* of the particle's kinetic and potential energies should remain unchanged. Indeed, Eq. (B-3) implies that

$$K_b + U_b = K_a + U_a \quad (\text{B-4})$$

so that the sum $K + U$ has always the same value.

Let us then introduce the convenient abbreviation E , called simply the “energy” (or sometimes the “total energy”) of the particle.

$$\text{Def. } \boxed{\text{Energy: } E = K + U} \quad (\text{B-5})$$

In other words, the energy of the particle due to specified forces is the sum of the kinetic energy of the particle plus its potential energy due to these forces. Then the result Eq. (B-4) can be summarized by the statement that

$$\boxed{E = K + U = \text{constant}} \quad (\text{B-6})$$

where the energy E is due to *all* forces which do work on the particle.

The relation (B-6) expresses the “principle of conservation of energy” and can be stated in words:

Conservation of energy: If the work done by every force which does work on a particle is independent of the process, the energy of the particle due to all these forces remains constant.

(B-7)

The properties of the energy $E = K + U$ follow from those of the kinetic energy K and potential energy U . Thus the energy E is a number having the unit “joule.” Furthermore, since the potential energy can have any *sign* (positive, negative, or zero) and has a value depending on the choice of a standard position, the energy E can also have any sign and also depends on the standard position used for specifying the potential energy. Note that the kinetic energy of a given particle depends only on its speed, while the potential energy of the particle depends only on its position (for a fixed choice of the standard position). Hence the energy E of the particle depends on *both* its position and its velocity (i.e., it depends on the *state* of the particle). As the particle moves, its kinetic energy changes because the particle's speed changes, and its potential energy changes because the particle's position changes. But, if the work done by every force which does work on the particle is independent of process, the conservation of energy implies that the sum $E = K + U$ remains unchanged.

Example B-1: Motion of a pendulum

A pendulum consists of a particle attached to a light string suspended from the ceiling. As the particle swings back and forth, what is the relationship between its speed and position?

The situation is illustrated in Fig. B-1. The particle is acted on by the gravitational force \vec{F}_g due to the earth and by the tension force \vec{F}_t due to the string. Since this tension force is along the string and thus always perpendicular to the particle's path, it does no work on the particle. Hence the only work done on the particle is that done by the gravitational force \vec{F}_g . But since this gravitational force is constant, the work done by it is independent of the process. Hence the particle has a potential energy which is due solely to the gravitational force. Consequently we can use the principle of conservation of energy to find a relation between the particle's kinetic and potential energies, and thus to find a relation between its speed and position.

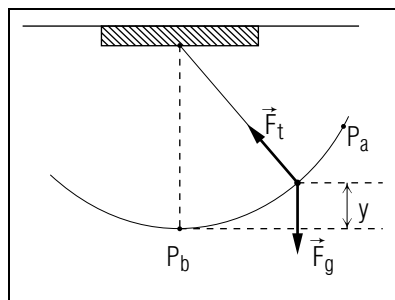


Fig. B-1: Motion of a pendulum.

Suppose that the particle has a mass m . If the particle's speed is v , its kinetic energy is $K = 1/2mv^2$. If we choose the standard position of the particle at the lowest point of its swing and measure its height y vertically upward above this point, the particle's gravitational potential energy is $U = mgy$. Hence the conservation of energy implies that

$$E = K + U = \frac{1}{2}mv^2 + mgy = \text{constant} \quad (\text{B-8})$$

Let us first look at the situation qualitatively. When the particle is momentarily at rest at the highest point of its swing, its kinetic energy K is zero (since $v = 0$) and its potential energy U is maximum (since the height y of the particle is maximum). As the particle descends along its path so that its height decreases, its potential energy U *decreases*. Since the energy $E = K + U$ of the particle remains constant, the particle's kinetic energy K , and thus also its speed v , must then *increase*. At the lowest point of its path, the particle's potential energy U is minimum. Since the energy E remains constant, the particle's kinetic energy K , and thus its speed v , must then be maximum. When the particle then moves up again along its path, its potential energy U increases. Hence its kinetic energy, and thus also its speed, decreases.

We can readily discuss the particle's motion quantitatively by comparing the particle in two different states a and b . The conservation of energy then implies that $E_a = E_b$ or

$$K_a + U_a = K_b + U_b \quad (\text{B-9})$$

For example, suppose that the particle is initially at rest at its highest point P_a where its velocity $v_a = 0$. What then is the speed of the particle at the lowest point P_b of its path? In this case, $K = 0$ since $v_a = 0$. Also $U_b = 0$ since $y_b = 0$ (i.e., because the height y of the particle above its

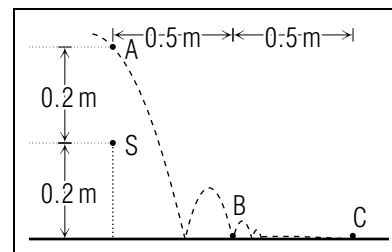


Fig. B-2.

lowest point is then zero). Hence the conservation of energy Eq. (B-9) implies simply

$$U_a = K_b$$

or

$$mgy_a = \frac{1}{2}mv_b^2 \quad (\text{B-10})$$

Dividing both sides of this last equation by m , we then obtain

$$v_b^2 = 2gy_a$$

or

$$v_b = \sqrt{2gy_a} \quad (\text{B-11})$$

Understanding the Definition of Energy (Cap. 1b)

B-1 *Statement and example:* We can specify the position of a ping-pong ball relative to a standard position chosen at the point S in Fig. B-2 by giving the ball's height y above the point S and its horizontal distance x to the right of this point. (a) If the ball has a mass m and speed v , express its energy E due to the gravitational interaction in terms of the symbols provided. (b) Suppose that $m = 0.02$ kg. What is the value of the ball's energy E for each of the states listed in Fig. B-2? (*Answer: 112*)

State	Position	Speed
a	Point A	2 m/s
b	Point B	2 m/s
c	Point C	0

B-2 *Properties:* (a) List the following properties of the quantities work, kinetic energy, potential energy, and energy: kind of quantity (number or vector), possible signs of numerical quantities, single SI

unit. (b) Which properties *differ* for these quantities? (c) Which quantities have values which depend on the choice of the standard position? (Answer: 109)

B-3 *Comparing forms of energy:* When a ball is in a state s , it is located at a point P_s and has a velocity \vec{v}_s . For each of the following different states of the ball, indicate whether the ball's kinetic energy K , gravitational potential energy U , and energy E are the same as or different than its corresponding energies K_s , U_s , and E_s in the state s . (a) The ball is located at the same point P_s , but it has a different speed and thus a different velocity. (b) The ball is located at the same point P_s and it has the same speed, but its direction of motion, and thus its velocity, is different. (c) The ball is located at a point higher than P_s , but its velocity is the same. (Answer: 114)

Understanding Conservation of Energy (Cap. 1c)

B-4 *Example:* A 0.50 kg stone, initially at rest at a point P , falls vertically downward with negligible air resistance so that only the earth's gravitational force does work on the stone. (a) If we choose the point P as the standard position, what is the stone's initial energy E due to the gravitational interaction? (b) When the stone has fallen 5.0 meter from the point P , what is its energy E ? What is its gravitational potential energy? What is its kinetic energy? (c) *Review:* What is the stone's speed when it has fallen 5.0 meter? (Answer: 107)

B-5 *Meaning of U :* We can see from the motion of each of the following particles that the particle's energy $E = K + U$ (where U is its gravitational potential energy) is *not* conserved. By examining the forces doing work on each particle, briefly explain why this energy is not conserved. (a) A sky diver falls with constant speed because of an upward force on her due to air resistance. (b) A box slides down a ramp with decreasing speed because of the frictional force exerted on it by the ramp surface. (c) A hockey puck accelerates across a level frictionless ice surface because of the force exerted on it by a hockey stick. (Answer: 116)

B-6 *Dependence:* When a positively-charged particle is scattered by a nucleus, only the repulsive Coulomb electric force² \vec{F}_e due to the nucleus does work on the particle. This work is independent of process.

²Note that no understanding of the Coulomb electric force is needed for the problem at hand.

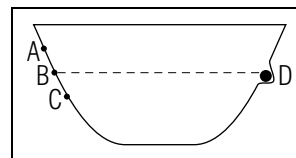


Fig. B-3.

The particle's energy $E = K + U$, where U is its potential energy due to the force \vec{F}_e , is thus conserved. (a) As the particle approaches the nucleus, its speed and kinetic energy decrease. What happens to its potential energy? (b) As the particle travels away from the nucleus, its potential energy decreases. What happens to its kinetic energy and speed? (Answer: 113)

Now: Go to tutorial section B.

Relating Speed and Position (Cap. 3)

B-7 The simple game in Fig. B-3 consists of a smooth bowl with a small depression D in the right side, and a marble which rolls in the bowl with negligible friction. The aim is to start the marble on the left in such a way that it arrives at D with zero speed and is trapped. To win, at which point (A , B , or C) should one start the marble (a) by releasing it at rest, and (b) by giving it a small initial speed? (Answer: 119) (Suggestion: [s-8])

B-8 An electron of mass 1×10^{-30} kg moves from a point A to a point B under the sole influence of an electric force \vec{F}_e due to the charged deflection plates of an oscilloscope tube (see problem A-5). The electron's potential energy due to \vec{F}_e is $+3 \times 10^{-18}$ J at the point A , and is -7×10^{-18} J at the point B . If the electron's speed at the point A is 9×10^6 m/s, what is its speed at the point B ? (Answer: 111) (Suggestion: [s-3])

B-9 A beginner's ski slope is designed so that, even if an 80 kg skier starts at the top of the slope with a speed of 1.0 m/s and skis with negligible friction, his speed at the bottom of the slope will only be 9.0 m/s. Use this information to find the vertical distance between the bottom and top of the slope. (Answer: 117) (Suggestion: [s-9])
TxtCapPrac[p-1], [p-2]

SECT.

C WORK DONE BY CENTRAL FORCES

As we have seen, the conservation of energy applies whenever the work done by the forces on a particle is independent of the process (meaning independent of the path over which the work is done). In the special case where the force on a particle is constant throughout some region, the work is certainly independent of the process (as discussed in Sec. A). But ordinarily the forces on a particle depend on its position and are thus not constant. In particular, consider the force \vec{F} on a particle due to another particle fixed at a point P_0 . Such a force is ordinarily a “central force” [as defined in statement (E-1) of Unit 408] which has a direction along the line joining the particles and has a magnitude depending only on the distance R between the particles. How does the work done on the particle by such a central force \vec{F} depend on the process?

Let us first find the work $\delta W = F dr_F$ done on the particle when it moves through a small enough displacement $d\vec{r}$ from some point P_1 , at a distance R_1 from P_0 to another point P_2 , at a distance R_2 from P_0 . (See Fig. C-1a.) The magnitude F of the force \vec{F} on the particle at P_1 depends only on the distance of P_1 from P_0 . The numerical component dr_F of the displacement $d\vec{r}$ along \vec{F} (i.e., along the radial line joining the points P_0 and P_1) is just the distance from P_1 to the point P_3 which lies on this radial line at the same distance R_2 as the point P_2 . *

* Since $d\vec{r}$ is small enough, the line from P_2 to P_3 is perpendicular to the force \vec{F} directed along the line joining P_0 and P_1 .

Hence dr_F is just equal to the difference $R_2 - R_1$ between the distances R_2 and R_1 of the points P_2 (or P_3) and P_1 from P_0 . Thus we see that the work $\delta W = F dr_F$ done on the particle in a small enough displacement between two points depends only on the distances R_1 and R_2 of these points from the position P_0 of the fixed particle. Hence the same work is done on the particle in a small displacement from *any* point at a specified distance R_1 from P_0 to *any* other point at a specified distance R_2 from P_0 . (For example, in Fig. C-1a the same work is done between the points P'_1 and P'_2 as between the points P_1 and P_2 .)

Consider now the work done on the particle along two different paths between any two points P_a and P_b . (See Fig. C-1b.) Then we need only add the works done in successive small enough displacements along each

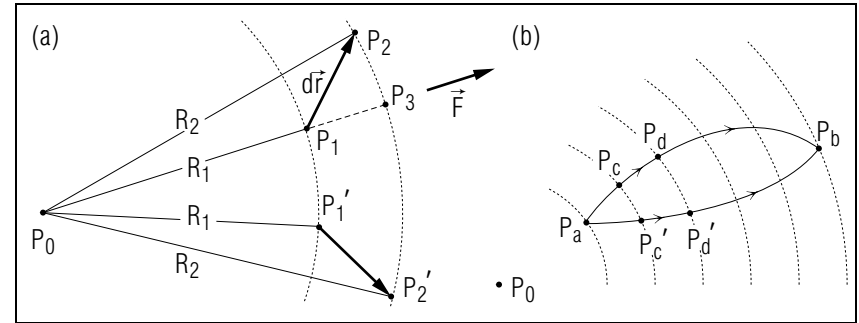


Fig. C-1: Work done on a particle by a central force due to a particle fixed at P_0 . (a) Small displacement. (b) Displacements along two different paths between the same points.

of these paths. But the work done from P_a to P_c is the same as that done from P_a to P'_c (since P_c and P'_c are at the same distance from P_0); the work done from P_c to P_d is the same as that done from P'_c to P'_d ; and so on. Hence the work done along the entire path from P_a to P_b is the same for each of the two paths. Since this statement holds for *any* two paths between P_a and P_b , we conclude that *the work done on a particle by a central force is independent of the process.*

More generally, the particle may be acted on by central forces due to *several* particles at fixed positions. The work done on the particle in any process is then, by Relation (D-5) of Unit 414, equal to the sum of the works done on the particle by the individual central forces. But since each of these individual works is independent of the process, the total work done by several central forces acting jointly must also be independent of the process. Thus we arrive at this conclusion:

The work done on a particle by central forces due to fixed particles is independent of the process. (C-1)

Most fundamental forces, such as the gravitational force or the electric Coulomb force, are central forces. Even if they are not, the conclusion in Rule (C-1) still holds. Thus *the work done on a particle by fundamental forces due to fixed particles is always independent of the process.*

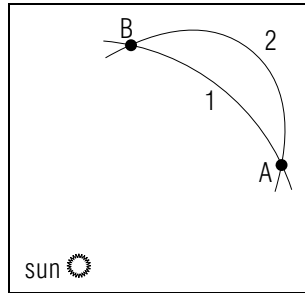


Fig. C-2.

Knowing About Work Done by Central Forces

C-1 A meteor moves from a point A to a point B under the sole influence of the gravitational force \vec{F}_g due to the sun. The meteor can move with constant speed along the short circular path 1 shown in Fig. C-2, or it can move with varying speed along the longer path 2. Zero work is done on the meteor by \vec{F}_g along the short path 1. Is the work done by \vec{F}_g along the longer path 2 larger than, equal to, or smaller than zero? (*Answer: 121*)

SECT.

D POTENTIAL ENERGY DUE TO CENTRAL FORCES

Consider a particle acted on by a central force due to another particle fixed at a point P_0 . Since the work done on the particle is independent of the process, we know from Sec. A that we can find this work from a potential energy by using the relation $W_{ab} = U_a - U_b$. Here the potential energy U_a at any point P_a is defined as $U_a = W_{as}$, where W_{as} is the work done by the force when the particle moves from P_a to some standard position P_s . Unless explicitly stated otherwise, we shall find it most convenient to choose this standard position P_s to be some point far enough from P_0 so that the interaction of the particle with the particle at P_0 is negligible.

REMARK ON EQUIVALENT STANDARD POSITIONS

It does not matter *which* point we choose as standard position, as long as it is far enough from P_0 . Indeed, suppose that we chose as standard position the point P'_s instead of the point P_s . (See Fig. D-1.) Then $W_{as'} = W_{as} + W_{ss'}$, since the work done from P_a to P'_s , being independent of the process, can be calculated as the particle first moves from P_a to P_s and then from P_s to P'_s . But $W_{ss'} = 0$ since the path from P_s to P'_s is so far from P_0 that the force on the particle is negligible. Hence $W_{as'} = W_{as}$, or $U'_a = U_a$. Thus the value of the potential energy U is the same for *any* choice of standard position far enough from P_0 .

How does the potential energy U of the particle depend qualitatively on its distance from the particle with which it interacts? Suppose that the particle is at a point P_a at a distance R_a from P_0 . Then U_a is the work W_{as} done on the particle when it moves from P_a to a standard position P_s very far away. (See Fig. D-2.) If the force on the particle has always the same direction (either repulsive or attractive), the *magnitude* of the work W_{as} is smaller if the distance moved by the particle from P_a to P_s is smaller, i.e., if the point P_a is farther from P_0 . Hence the *magnitude* of the potential energy $U_a = W_{as}$ of the particle is smaller if the distance R_a of the particle from P_0 is larger. In particular, $U_a = 0$ if the particle is very far from P_0 (so that negligible work is done when the particle moves to P_s).

The *sign* of the potential energy depends on the direction of the force. If the force on the particle is repulsive, the work W_{as} done on the particle in moving from P_a to P_s is positive since the direction of the force is then

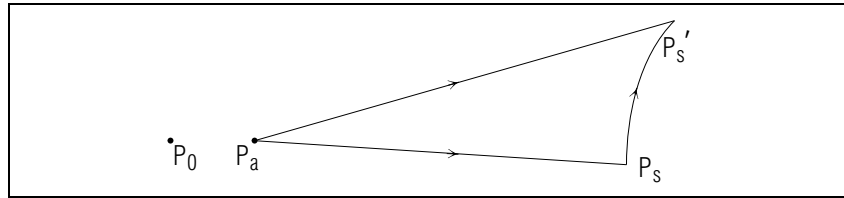


Fig.D-1: Potential energy calculated with respect to two different distant standard positions P_s and P'_s .

along the direction of motion of the particle. Hence $U_a = W_{as}$ is positive. Conversely, if the force on the particle is attractive, the work W_{as} done on the particle in moving from P_a to P_s is negative since the direction of the force is then opposite to the direction of motion of the particle. Hence $U_a = W_{as}$ is then negative.

To summarize, the *magnitude* of the potential energy U of the particle decreases with increasing distance R between the interacting particles, being positive if the force is repulsive and negative if it is attractive.

As indicated in Fig. D-3, this means that the potential energy U decreases with increasing distance if the force is repulsive, and that U increases (i.e., becomes less negative) with increasing distance when the force is attractive.

Example D-1: Gravitational potential energy due to the earth earth

As we know from text section A of Unit 410, the gravitational force on a particle (such as a satellite) due to the spherical earth is the same as if the entire mass of the earth were concentrated in a point at its center. Since this force is attractive, the potential energy of the particle due to its gravitational interaction with the earth is then negative and has a *magnitude* which decreases with increasing distance from the earth (as illustrated in Fig. D-3b). In other words, the potential energy then *increases* (i.e., becomes less negative) with increasing distance from the earth.

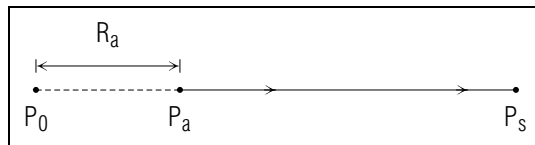


Fig.D-2: Calculation of the potential energy at the point P_a .

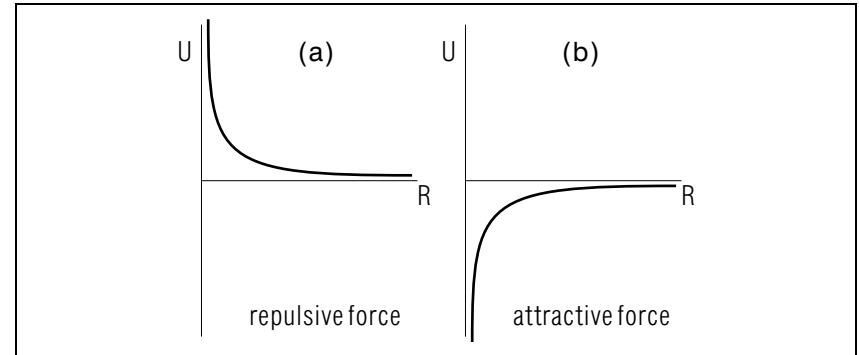


Fig. D-3: Potential energy due to repulsive and attractive forces.

Let us compare this general result with our previous special discussion of the gravitational potential energy *near the surface* of the earth. In that case we chose as the standard position a point at the surface of the earth (rather than a point very far from the earth) We then found in Eq. (A-7) that, close to the earth, $U = mgy$ where y is the height of the particle above the surface of the earth. Thus U increases with increasing height y of the particle. Note that this conclusion, valid near the surface of the earth, agrees with our general result that the gravitational potential energy increases with increasing distance from the earth.

POTENTIAL ENERGY FOR “INVERSE-SQUARE” FORCES

Both the gravitational force and the electric Coulomb force on a particle depend on the distance R from the particle with which it interacts so that the magnitude F of this force is

$$F = \frac{c}{R^2} \tag{D-1}$$

where c is a constant that is different for the gravitational and electric Coulomb forces. As we show in Section H, the potential energy U of the particle located at a distance R from the other particle is then

$$U = \pm \frac{c}{R} \tag{D-2}$$

where the + sign applies if the force is repulsive and the – sign applies if the force is attractive. (Note that the potential energy U is inversely proportional to R , unlike the force F which is inversely proportional to R^2 .)

For example, the gravitational force on a particle of mass m_1 , interacting with another particle of mass m_2 , has a magnitude $F = Gm_1m_2/R^2$ (see Appendix page A6-1 for the value of the universal constant G). Hence Eq. (D-2) implies the following result for the potential energy of the particle due to this force:

$$\boxed{\text{Gravitational potential energy: } U = -\frac{Gm_1m_2}{R}} \quad (\text{D-3})$$

where the minus sign is appropriate since the gravitational force is attractive. As another example, the electric Coulomb force on a particle with charge q_1 , interacting with another particle with charge q_2 , has a magnitude $F = k_e|q_1||q_2|/R^2$ (see Appendix page A6-1 for the value of the universal constant k_e). Hence Eq. (D-2) implies the following result for the potential energy of the particle due to this force:

$$\boxed{\text{Coulomb potential energy: } U = k_e\frac{q_1q_2}{R}} \quad (\text{D-4})$$

If both charges have the same sign, the Coulomb force is repulsive and U is properly positive (since the product q_1q_2 is then positive). If the charges have opposite signs, the Coulomb force is attractive and U is properly negative (since the product q_1q_2 is then negative).

Note that the quantitative results in Rule (D-3) and Rule (D-4) agree with the qualitative graphs of Fig. D-3.

POTENTIAL ENERGY DUE TO SEVERAL PARTICLES

Suppose that a particle interacts by central forces with *several* other particles. As mentioned at the end of the preceding section, the work done on the particle is then also independent of the process so that it can be found from a potential energy. As usual, this potential energy is just the work done on the particle when it moves to the standard position. But, by Relation (D-5) of Unit 414, this work is merely the sum of the works done on the particle by the individual forces separately. Hence we arrive at this conclusion:

$$\boxed{\text{The potential energy of a particle interacting with several particles is the sum of the individual potential energies of the particle interacting with these particles separately.}} \quad (\text{D-5})$$

Relating Potential Energy and Position (Cap. 2b)

D-1 (a) What is the gravitational potential energy of the earth due to its interaction with the sun? Use the values 2.0×10^{30} kg and 6.0×10^{24} kg for the masses of the sun and earth, 1.5×10^{11} meter for the distance between the sun and the earth, and $G = 6.7 \times 10^{-11}$ N m²/kg². (b) What is the Coulomb potential energy of one nitrogen nucleus in the nitrogen molecule N_2 due to its interaction with the other nitrogen nucleus? Use the approximate values 2×10^{-18} C for the charge of a nitrogen nucleus, where C is the abbreviation for a coulomb of charge, and use 1×10^{-10} meter for the distance between the nuclei, and note that $k_e = 9 \times 10^9$ N m²/C². (c) What is the Coulomb potential energy of a nitrogen nucleus due to another located so far away that the interaction between the nuclei is negligible? (*Answer: 115*)

D-2 Consider the Coulomb potential energy U of a particle with charge q due to its interaction with a fixed particle of charge Q . For each of the following possible signs of q and Q , indicate which of the graphs in Fig. D-3 describes how U varies with the distance R between the particles. Then answer this question: Is the value of U when the particles are close together larger or smaller than the value of U when they are farther apart? (a) q and Q are both positive, (b) q and Q are both negative. (c) q is positive but Q is negative, (d) q is negative but Q is positive. (*Answer: 124*) (*Suggestion: [s-4]*)

D-3 (a) Fig. D-4a shows the path of a positively-charged alpha particle interacting only by the Coulomb electric force with a positively-charged nucleus. At which of the points indicated is the alpha particle's Coulomb potential energy largest? At which of these points is it smallest? At which pair of points is it the same? (b) Fig. D-4b shows the path of a comet interacting only by the gravitational force with the sun. At which of the indicated points is the comet's gravitational potential energy largest? At which of these points is it smallest? (c) *Review:* At which of the indicated points is the alpha particle's kinetic energy and speed largest? At which of the indicated points is the comet's kinetic energy and speed largest? (*Answer: 120*) (*Suggestion: [s-11]*)

SECT.

E

 CONSERVATION OF ENERGY FOR CENTRAL FORCES

Our previous discussion shows that, whenever a particle is acted on by central forces, the work done on the particle is independent of the process and can be found by using potential energy. Hence the principle of conservation of energy of Sec. B applies under these general conditions. Since fundamental forces, such as the gravitational and Coulomb forces, are central forces, the principle of conservation of energy applies then whenever a particle is acted on only by such fundamental forces due to other fixed particles.

Example E-1: Closest approach of a proton to a nucleus

If protons can be shot at a nucleus so as to approach this nucleus sufficiently closely, the scattering of the protons can provide valuable information about the structure of the nucleus. Suppose that a proton, having a mass m and a charge $+e$, is shot with an initial speed v_a at such a nucleus. If the nucleus has a charge Q and a mass much larger than that of the proton, how close can the proton come to the nucleus despite the repulsive Coulomb force on the proton due to the nucleus?

Description: Fig. E-1 shows the situation. Here P_a is a point far from the nucleus where the proton has its initial speed v_a and P_b is the point where the proton is closest to the nucleus.

Planning: Since the nucleus has a mass much larger than that of the proton, its acceleration is negligible so that it can be considered fixed relative to an inertial frame. The proton is only acted on by the repulsive Coulomb force due to this fixed nucleus. Its Coulomb potential energy U due to this interaction is then positive and decreases with increasing distance from the nucleus. (See Fig. D-3a.)

To understand what happens, let us first look at the situation qualitatively. When the proton is at the point P_a very far from the nucleus, its potential energy $U_a = 0$ while its kinetic energy is $K_a = 1/2mv_a^2$ since its

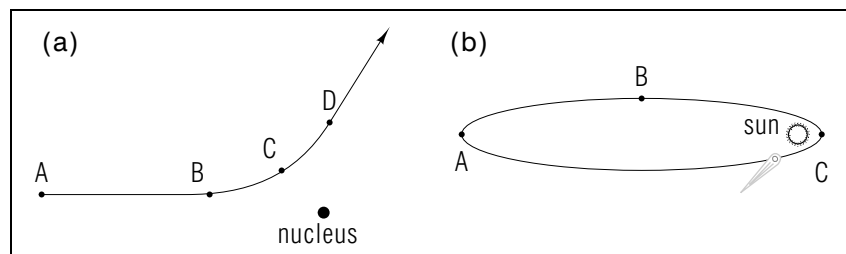


Fig. D-4.

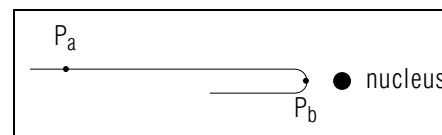


Fig. E-1: Approach of a proton to a nucleus in a nearly head-on collision.

initial speed is v_a . The initial energy of the proton consists then purely of its kinetic energy K_a . As the proton comes closer to the nucleus, its Coulomb potential energy U increases. Since the conservation of energy implies that the energy $E = K + U$ of the proton remains constant, the kinetic energy K (and hence also the speed v) of the proton must then decrease. The minimum possible value of the kinetic energy is zero and is attained if the proton is at some point P_b where its speed $v_b = 0$. (See Fig. E-1.) Since the energy $E = K + U$ remains constant, the potential energy U_b of the proton must then have its maximum possible value, i.e., the proton must then be as close to the nucleus as possible.

To discuss the situation quantitatively, we need only compare the energy of the proton in its initial state a far from the nucleus with its energy in its state b closest to the nucleus. Since these energies must be equal,

$$K_a + U_a = K_b + U_b$$

or

$$K_a = U_b \quad (\text{E-1})$$

because $U_a = 0$ initially and $K_b = 0$ finally. Here $K_a = 1/2mv_a^2$ and U_b is the Coulomb potential energy given by Rule (D-4) so that $U_b = k_e eQ/R_b$.

Implementation: By using the expressions for K_a and U_b in Eq. (E-1) we obtain

$$\frac{1}{2}mv_a^2 = k_e \frac{eQ}{R_b} \quad (\text{E-2})$$

or

$$R_b = \frac{2k_e eQ}{mv_a^2} \quad (\text{E-3})$$

which tells us the closest possible distance R_b of the proton from the nucleus.

Checking: We would expect that the proton can come closer to the nucleus, despite the repulsive Coulomb force, if the initial speed of the proton is larger. The result Eq. (E-3) agrees with this expectation since it implies that the distance R_b is smaller if the initial speed v_a is larger.

Relating Speed and Position (Cap. 3)

E-1 Near a uranium nucleus in an electron-microscope sample, an electron is scattered by a total force equal to the Coulomb electric

force due to the nucleus. Is the electron's initial speed at a point A far from the nucleus larger than, equal to, or smaller than its speed at a point C when it is closest to the nucleus? After scattering from the nucleus, the electron passes a point E the same distance from the nucleus as the point A . Is the electron's speed at the point E larger than, equal to, or smaller than its speed at the point A ? (*Answer: 123*) (*Suggestion: [s-7]*)

E-2 Halley's comet travels along a very long, thin elliptical orbit, so that its speed is negligible when it is located at the point A (called the "aphelion") at its farthest distance R_a from the sun. Although the comet cannot be observed at the point A , its speed v_b can be measured when it is located at a point B at a smaller distance R_b from the sun. We can assume that this comet interacts only with the sun, which has a known mass M . Write an equation relating the unknown distance R_a to the known quantities M , R_b , and v_b . (You need not solve this equation for R_a .) (*Answer: 118*) (*Suggestion: [s-5]*) *More practice for this Capability: [p-3], [p-4]*

SECT.

F SUMMARY

DEFINITIONS

potential energy; Def. (A-3)

energy; Def. (B-5)

IMPORTANT RESULTS

Definition and properties of potential energy: Def. (A-3), Rule (A-5)

$$U_a = W_{as}, U_s = 0$$

Relation between work and potential energy: Eq. (A-4)

$$\text{If work is independent of process, } W_{ab} = U_a - U_b$$

Conservation of energy: Rule (B-7)

If work due to all forces is independent of process,

$$E = K + U = \text{constant}$$

Gravitational potential energy near the surface of the earth: Eq. (A-7)

$$U = mgy$$

Work and potential energy due to central forces: Rule (C-1), Rule (D-3), Rule (D-4)

Work is independent of process.

$$\text{Gravitational potential energy: } U = -Gm_1m_2/R$$

$$\text{Coulomb potential energy: } U = k_e q_1 q_2 / R$$

NEW CAPABILITIES

You should have acquired the ability to:

- (1) Understand these relations for a single particle:
 - (a) the relation $W_{ab} = U_a - U_b$ (Sec. A),
 - (b) the definition of energy $E = K + U$ (Sec. B),
 - (c) the principle of conservation of energy (Sec. B).
- (2) Qualitatively or quantitatively relate a particle's position and its potential energy due to:
 - (a) the gravitational force near the earth (Sec. A),
 - (b) a gravitational or Coulomb electric force due to a fixed particle (Sec. D).
- (3) Apply conservation of energy to relate, quantitatively or qualitatively, a particle's position and speed. (Sects. B and E, [p-1] through [p-4])

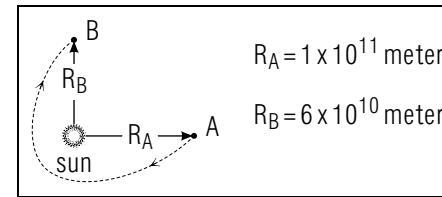


Fig. F-1.

Applying Work and Energy Relations (Cap. 1, 2, 3)

F-1 The oriental rat flea *Xenopsylla cheopis*, of mass 2×10^{-7} kg, can jump vertically upward with an initial speed of 1.4 m/s. In such a jump, the flea is momentarily at rest at a maximum height of 9.0 cm above its original position. To illustrate the effects of air resistance on the flea, find the flea's maximum height if it jumps upward with the same initial speed but moves with negligible air resistance. (*Answer: 127*)

F-2 The work done on a particle by a central force can be found much more easily by using potential energy than by computing the sum of small works done along the particle's path. To illustrate, consider a 300 kg space probe which moves along the path in Fig. F-1 from a point A on the orbit of Venus to a point B on the orbit of Mercury. What is the work done on the probe by the gravitational force due to the sun? Use the values $M = 2 \times 10^{30}$ kg for the sun's mass and $G = 7 \times 10^{-11}$ N m²/kg². (*Answer: 125*)

F-3 Suppose that a particle of mass m is launched with some initial speed from the surface of a planet of mass M and radius R , and thereafter moves under the sole influence of the planet's gravitational force. If the particle's initial speed is equal to the so-called "escape velocity" v_e for the planet, the particle will arrive at a point very far from the planet with negligible speed (so that it barely "escapes" and never returns to the planet's vicinity). (a) Use this information to express the escape velocity v_e in terms of the quantities m , M , and R . (b) What is the escape velocity v_e for the earth? Use the values $M = 6.0 \times 10^{24}$ kg, $R = 6.4 \times 10^6$ meter, and $G = 6.7 \times 10^{-11}$ N m²/kg². (*Answer: 122*)

Organizing Relations (Cap. 1)

F-4 A toboggan slides down a snowy hill, and a student wants to find its final speed by applying the relation between work and kinetic energy or the principle of conservation of energy. (a) If the frictional force on the toboggan due to the snow is negligible, which of these relations can he use? (b) Which of these relations can he use if this frictional force is *not* negligible? (*Answer: 128*)

SECT.

G PROBLEMS

G-1 *Using energy to compare projectile motions:* A boy throws a stone twice with the same initial speed and from the same point. The first time, he throws the stone vertically upward, so that when it reaches its maximum height above the ground it is momentarily at rest. The second time, he throws the stone at an angle to the ground, so that it is moving horizontally when it reaches its maximum height above the ground. In both throws, the stone moves with negligible air resistance. (a) Is the initial value of the stone's energy due to the gravitational interaction the same or different for the two throws? (b) Is this energy conserved during each throw? (c) Is the maximum height for the second throw larger than, equal to, or smaller than that for the first throw? (*Answer: 130*)

G-2 *Speed of closest approach in electron scattering:* As an electron of mass 1.0×10^{-30} kg approaches a uranium nucleus in an electron-microscope sample, it moves under the sole influence of the Coulomb electric force exerted by the nucleus. If the electron's initial speed when it is far from the nucleus is 5.0×10^7 m/s, what is its speed at its point of closest approach, 1.0×10^{-10} meter from the nucleus? Use the values 3.8×10^{-17} coulomb and 1.6×10^{-19} coulomb for the magnitudes of the charges of the nucleus and the electron. (*Answer: 126*)

G-3 *Escape velocity for the solar system:* Suppose a particle is launched from the surface of a planet, of mass M and radius R , which is moving in an orbit of radius R_0 about the sun, of mass M_s . If the particle has a large enough initial speed at the planet's surface, it will barely "escape" from the solar system by moving under the joint influence of the planet and the sun to a point far from both, where its speed is negligible. (a) Find an expression for this initial speed (or "escape velocity") V_e . (b) Is V_e larger than, equal to, or smaller than the escape velocity v_e found in problem F-3 for the planet alone? (c) Find the value of V_e for the earth, using the values listed in problem F-3 and the values $M_s = 2.0 \times 10^{30}$ kg and $R_0 = 1.5 \times 10^{11}$ meter. (*Answer: 129*) (*Suggestion: [s-1]*)

SECT.

H

POTENTIAL ENERGY FOR INVERSE-SQUARE-LAW FORCES

Suppose that the force \vec{F} on a particle due to a fixed particle at P_0 depends on the distance R between the particles so that $F = c/R^2$ where c is some constant. What then is the potential energy U of the particle at any point?

By its definition, the potential energy U_a at some point P_a at a distance R_a from P_0 is $U_a = W_{as}$ where W_{as} is the work done by the force on the particle as it moves from P_a to a standard position P_s very far from P_0 . (See Fig. D-5.) Assume that the force \vec{F} is repulsive so that it is directed away from P_0 . Then the work done on the particle in any small displacement from any point P (at a distance R from P_0) to a neighboring point P' (at a slightly larger distance $R' = R + dR$ from P_0) is

$$\delta W = F dR = \frac{c}{R^2} dR \quad (\text{H-1})$$

We can then find the work W_{as} done along the entire straight path from P_a to P_s by calculating the sum of all such successive small works, starting at the point P_a and ending at the point P_s . In other words,

$$U_a = W_{as} = \int_a^s \delta W \quad (\text{H-2})$$

The preceding sum could be easily calculated if δW were merely equal to a small change in some quantity. But (1) shows that δW is not directly expressed as a small change, being equal to the small change dR multiplied by the quantity c/R^2 . But we can relate δW to a small change in the quantity c/R . Indeed, a small change in this quantity, corresponding to the small change $dR = R' - R$, is

$$\begin{aligned} d\left(\frac{c}{R}\right) &= \frac{c}{R'} - \frac{c}{R} = \frac{c}{R+dR} - \frac{c}{R} \\ &= \frac{cR - c(R+dR)}{(R+dR)R} = -\frac{cdR}{(R+dR)R} = -\frac{cdR}{R^2} \end{aligned}$$

since $R + dR$ in the denominator can be replaced by R when dR is small enough to be negligible compared to R . Thus

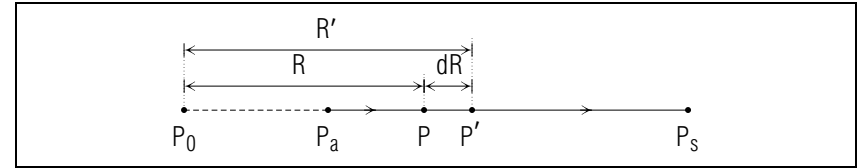


Fig. D-5: Calculation of work done on a particle moving from P_a to the distant standard position P_s .

$$\delta W = \frac{c}{R^2} dR = -d\left(\frac{c}{R}\right) \quad (\text{H-3})$$

Thus we obtain

$$W_{as} = -\int_a^s d\left(\frac{c}{R}\right) = -\left[\frac{c}{R_s} - \frac{c}{R_a}\right] \quad (\text{H-4})$$

since the sum of the successive small changes in the quantity c/R is simply equal to the total change in this quantity between the initial value where $R = R_a$ and the final value where $R = R_s$. But since the distance R_s of the standard point from P_0 is supposed to be very large, c/R_s is negligibly small. Hence (4) yields

$$U_a = W_{as} = -\left[0 - \frac{c}{R_a}\right] = \frac{c}{R_a} \quad (\text{H-5})$$

In other words, the potential energy U of the particle at any point at a distance R from P_0 is simply $U = c/R$, as stated in Eq. (D-2). (If the force were attractive, the work W_{as} would simply have a negative sign so that U_a would correspondingly also have a negative sign.)

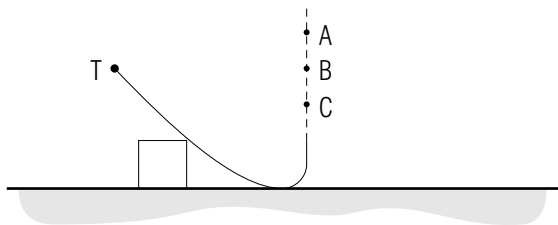
TUTORIAL FOR B

RELATING SPEED AND POSITION USING ENERGY CONSERVATION

b-1 *PURPOSE:* Whenever the forces doing work on a particle do work that is independent of process, we can use the principle of conservation of energy to relate the particle's speed and position at points along its path. To do so, we find all the forces doing work on the particle, and express the particle's energy $E = K + U$ in terms of a potential energy U due to these forces. Using this expression, and the fact that E has the same value at all points along the particle's path, we can find the desired information about the particle's speed or position at one point by using its known speed and position at another.

The purpose of the next two frames is to illustrate this basic method in more detail, first for a qualitative problem and then for a quantitative one. For both kinds of problem, we shall follow the basic steps of the problem-solving strategy outlined in text section D of Unit 409.

b-2 *A METHOD FOR QUALITATIVELY RELATING SPEED AND POSITION:* Let us systematically solve the following qualitative problem: Using a piece of smooth, thin metal, a boy makes a track to catapult his toy car vertically upward, as shown in the following drawing. He releases the car with a small initial speed at the top point T of the track, and the car moves with negligible air resistance and friction down the slope, up the other side, and into the air. At which of the indicated points does the car finally reach its maximum height, where it is momentarily at rest?



DESCRIPTION:

Sketch: we have one already. *Known information:* the car has a small speed at T , friction and air resistance are negligible. *Desired information:* point at which car is at rest at its maximum height.

PLANNING:

- Decide whether conservation of energy is applicable.*
To do so, we must identify forces doing work on the particle (i.e., the car), and be sure these forces do *work independent of process*. The forces due to friction and air resistance are negligible, and the normal force due to the track is everywhere perpendicular to the car's path. Thus the gravitational force alone does work on the car, and this force does work independent of process. Conservation of energy applies.
- Express the principle of conservation of energy in terms of known and desired information.*
Since the gravitational force alone does work on the car, the car's energy $E = K + U$, where U is its gravitational potential energy, is thus conserved. While K depends only on the car's speed, U depends only on its height. Using the known change in the car's speed between its initial and final states, we can thus find the unknown change in its height.

IMPLEMENTATION:

- Use known information to describe the change in K (or U) between the particle's initial and final states.*
Since the car is initially moving at the point T and finally at rest at its maximum height, its kinetic energy has decreased.
- Use energy conservation to describe the corresponding change in U (or K).*
Since $E = K + U$ is constant and K has decreased, U must have increased.
- Find the desired information.*
Since the car's gravitational potential energy has increased, its final position must be higher than its initial position at the point T . Thus it comes to rest at the point A .

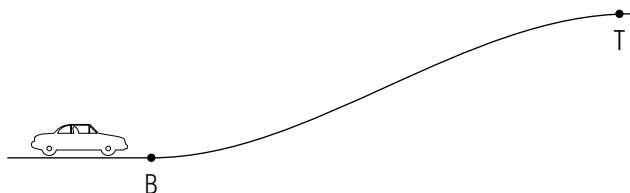
CHECKING:

Our argument shows correctly that the car's energy due to the gravitational interaction is conserved, since the increase in its potential energy compensates for the decrease in its kinetic energy.

b-3 *A METHOD FOR QUANTITATIVELY RELATING SPEED AND POSITION:* Let us systematically solve the following quantitative problem: Thick ice has formed on a road running up a hill 10 meter high, although the sun has melted the ice on the level road at the bottom of the hill. What speed must a 1500 kg car have at the bottom of the hill in order to barely arrive at the top with zero speed? (The frictional force due to the icy road surface is negligible, so the car has no traction on the hill.)

DESCRIPTION:

Sketch:



Known: the frictional force on the car is negligible on the hill, the hill is 10 meter high, the car's speed $v_t = 0$ at the top (T) of the hill, the car's mass $m = 1500$ kg. *Desired:* the car's speed v_b at the bottom (B) of the hill.

PLANNING:

- (1) *Decide whether conservation of energy is applicable.*
As in the last problem, only the gravitational force does work on the car, and this force does work independent of process. Thus conservation of energy applies.
- (2) *Express the energy conservation equation $K_a + U_a = K_b + U_b$ in terms of symbols for known and desired information.*
As a first step, we write $K_b + U_b = K_t + U_t$, where b and t indicate

the car's initial and final states at the bottom and the top of the hill. Since only the gravitational force does work on the car, U represents the car's gravitational potential energy. The next step is to choose a convenient standard position for U , and express each term in our equation in terms of symbols for known and desired quantities. Let us choose as a standard position the point B at the bottom of the hill, so that the car's initial potential energy $U_b = 0$. Since $v = 0$, the car's final kinetic energy $K_t = 1/2mv_t^2 = 0$. Our equation then becomes $1/2mv_b^2 + 0 = 0 + mgy_t$, where y_t is the car's final height at the top of the hill. Since m , g , and this height are known, we can find the desired speed v_b .

IMPLEMENTATION:

- (1) *Solve the equation algebraically for the desired quantity.*
Dividing our equation by m and multiplying by 2, we obtain

$$v_b^2 = 2gy_t, \text{ so that } v_b = \sqrt{2gy_t}.$$

- (2) *Substitute known values, and find the desired quantity.*
Since the hill is 10 meter high, $y_t = 10$ meter. Thus

$$v_b = \sqrt{2(10 \text{ m/s}^2)(10 \text{ meter})} = \sqrt{2.0 \times 10^2 \text{ m}^2/\text{s}^2}, \text{ or}$$

$$v_b = 1.4 \times 10^1 \text{ m/s} = 14 \text{ m/s}$$

CHECKING:

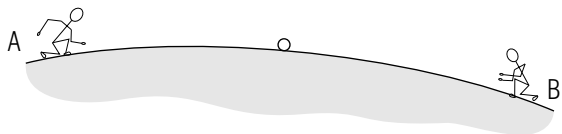
The work is correct, and the result has the correct unit and a reasonable magnitude. In addition, the result is qualitatively correct in that the car's kinetic energy has decreased while its potential energy has increased.

The methods we have illustrated are useful in applying conservation of energy to any problem. In particular, they should help you solve systematically the remaining problems in text section B.

Now: Go to text problem B-7.

PRACTICE PROBLEMS

p-1 *RELATING SPEED AND POSITION (CAP. 3):* Two children roll a 0.1 kg ball back and forth along the sidewalk as shown in the following drawing. They each give the ball the same initial speed of 3.0 m/s, and the ball rolls with negligible friction along the sidewalk. (a) Suppose the ball has a speed of 2.0 m/s just before it is caught. Which child catches the ball? (b) What is the vertical distance between the two children? (c) The ball is now rolled back the other way. What is its speed just before it is caught? (*Answer: 4*) (*Suggestion: review tutorial frames [b-2] and [b-3] or text problems B-7 through B-9. Further practice: [p-2].*)



p-2 *RELATING SPEED AND POSITION (CAP. 3):* To be effective in fighting a fire on the roof of a 7 story building, a water droplet directed upward from a fire hose at the base of the building must reach a height of at least 20 meter above the hose nozzle. By assuming that a droplet moves vertically upward with negligible air resistance, estimate the speed it must have when emerging from the nozzle in order to arrive at this height with zero speed. (*Answer: 1*) (*Suggestion: review tutorial frame [b-3] or text problems B-8 and B-9.*)

p-3 *RELATING SPEED AND POSITION (CAP. 3):* The planet Mercury moves along an elliptical (oval) orbit around the sun, and it interacts appreciably only with the sun. If Mercury's speed is larger at a point A than at a point B on its orbit, which of these points is closer to the sun? (*Answer: 3*) (*Suggestion: review text problem E-1 or tutorial frame [b-2]. Further practice: [p-2]*)

p-4 *RELATING SPEED AND POSITION (CAP. 3):* In a Van de Graaff accelerator, a particle of mass m and positive charge q is accelerated by a total force equal to the Coulomb electric force exerted by a metal sphere carrying a large positive charge Q . (This force, and the particle's corresponding Coulomb potential energy, are the same as those due to a particle of charge Q located at the center of the sphere.) Suppose

the particle is initially at rest at a point a distance R from the sphere's center, and then accelerates away from the sphere until it reaches a target so far from the sphere that the particle's interaction with the sphere is negligible. What is the speed v of the particle when it reaches the target? (*Answer: 2*) (*Suggestion: review text example E-1, text problem E-2, or tutorial frame [b-3].*)

SUGGESTIONS

s-1 (*Text problem G-3*): The particle's energy $E = K + U$, where U is its potential energy due to the gravitational interaction with *both* the planet and the sun, is conserved. Use statement D-4 in the text to find an expression for U .

s-2 (*Text problem A-1*): Part (a): Since the terminal B is the standard position, the electron's potential energy at terminal A is given by $U_a = W_{ab}$, the work done on the electron as it moves from terminal A to the standard position on terminal B . The electron's potential energy at the standard position on terminal B is just $U_b = W_{bb} = 0$.

s-3 (*Text problem B-8*): Use the method outlined in tutorial frame [b-3] as a guide. Your solution should include the following equation expressing the conservation of the electron's energy:

$$\frac{1}{2}mv_a^2 + U_a = \frac{1}{2}mv_b^2 + U_b$$

where m is the electron's mass, v_a and v_b are its speeds at the points A and B , and U_a and U_b are its potential energies at these points.

s-4 (*Text problem D-2*): In each case, first decide whether the Coulomb electric force is attractive or repulsive. When the force is repulsive, note that graph (a) in Fig. D-3 shows that the value of U is larger for small values of R (i.e., when the particles are close together) than it is for larger values of R (i.e., when the particles are farther apart). When the force is attractive, note that graph (b) in Fig. D-3 shows that the value of U is smaller (i.e., more negative) for small values of R than it is for larger values of R .

s-5 (*Text problem E-2*): Apply the principle of conservation of energy, using the method outlined in tutorial frame [b-3] as a guide. Since it interacts only with the sun, the comet's energy $E = K + U$, where U is its gravitational potential energy, is conserved. Thus we can write the equation $K_a + U_a = K_b + U_b$ expressing the equality of the comet's energy at the two points A and B . Express this equation in terms of the symbols provided. Note that the comet's unknown mass m appears in all terms of this equation, and thus may be eliminated by division.

s-6 (*Text problem A-6*): Suppose the particle's initial potential energy is U_a and its final potential energy is U_b . The work done as the particle moves from its initial to its final position is related to these potential energies by $W_{ab} = U_a - U_b$. Find the sign of W_{ab} by comparing the particle's direction of motion to the direction of the force. (You may want to review text section B of Unit 414.) If W_{ab} is positive, the particle's final potential energy U_b must be *smaller* than its initial potential energy U_a ; if $W_{ab} = 0$, U_b must be *equal* to U_a ; if W_{ab} is negative, U_b must be *larger* than U_a .

s-7 (*Text problem E-1*): Apply the principle of conservation of energy, following the method outlined in tutorial frame [b-2]. Since the only force on the negatively-charged electron is the attractive Coulomb force due to the positively-charged nucleus, the electron's energy $E = K + U$, where U is its Coulomb potential energy, is conserved. Use the fact that the nucleus exerts an *attractive* force on the electron to compare the electron's potential energy U at the positions described. (If you need more help, review text problems D-2 and D-3, or see suggestion [s-11].)

s-8 (*Text problem B-7*): Use the method outlined in tutorial frame [b-2] as a guide. Since the only force doing work on the marble is the gravitational force, the marble's energy $E = K + U$, where U is its gravitational potential energy, is conserved. In part (a), the marble's initial kinetic energy is *equal* to its desired final kinetic energy of zero. In part (b), the marble's initial kinetic energy is *larger* than its desired final kinetic energy. Use these observations to compare the marble's initial and final potential energies, and thus the marble's initial and final heights.

s-9 (*Text problem B-9*): Use the procedure outlined in tutorial frame [b-3] as a guide. Since the only force doing work on the skier is the gravitational force, the skier's energy $E = K + U$, where U is his gravitational potential energy, is conserved. For convenience, choose either the bottom or the top of the slope as the standard position. Your solution should then include one of the following equations expressing the conservation of the skier's energy (t and b indicate the skier's state at the top and bottom of the slope):

$$\text{Standard position at the bottom: } \frac{1}{2}mv_t^2 + mgL = \frac{1}{2}mv_b^2 + 0$$

$$\text{Standard position at the top: } \frac{1}{2}mv_t^2 + 0 = \frac{1}{2}mv_b^2 - mgL$$

(Note that these two equations are equivalent.)

s-10 (Text problem A-4): If a force does work which is independent of the process, the work W_{bc} done by this force as a particle moves from a point B to a point C depends *only* on the positions of the points B and C . Since $W_{bc} = U_b - U_c$, the difference $U_b - U_c$ between the particle's potential energies at these points also depends only on the positions of B and C . These quantities, therefore, do *not* change if we choose a different standard position for the particle's potential energy.

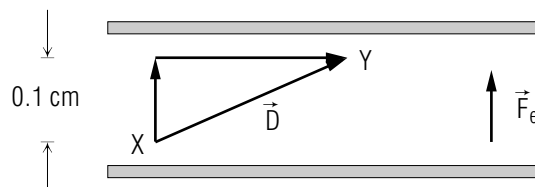
However, the particle's potential energy at a point is defined as the work done on the particle as it moves from this point *to the standard position*. Thus this potential energy *may* change if we choose a different standard position. For example, the baseball's potential energy at the point C is $U_c = 2\text{ J}$ if we choose the point A as the standard position. However, $U_c = 0$ if we choose the point C as the standard position.

s-11 (Text problem D-3): Parts (a) and (b): For each situation, decide whether the force acting on the particle is attractive or repulsive. Then compare the particle's potential energy at the points described by using the following characteristics of the potential energy U due to central forces (see Fig. D-3):

- (1) Attractive force: The *closer* the interacting particles are to each other, the *smaller* is the potential energy U .
- (2) Repulsive force: The *closer* the interacting particles are to each other, the *larger* is the potential energy U .
- (3) Either kind of force: Whenever the interacting particles are the *same* distance apart, the potential energy U is the *same*.

Part (c): Since each particle is acted on only by the central force described, its energy $E = K + U$, where U is the potential energy due to this force, is conserved. Thus the particle's kinetic energy is smallest when its potential energy U is largest, and vice versa.

s-12 (Text problem A-5): Part (a): As an example, let us find the work W_{xy} done on the electron as it moves from the point X to the point Y . Since the electric force \vec{F}_e is constant, we can use the relation $W = FD_F$ discussed in text section C of Unit 414. The following sketch shows the electron's displacement \vec{D} , and the component vectors of \vec{D} parallel and perpendicular to \vec{F}_e :



The sketch shows that the component D_F of \vec{D} along \vec{F}_e is positive (since \vec{D} is partly along \vec{F}_e) and has a magnitude of 1×10^{-3} meter. Thus

$$W_{xy} = FD_F = (1 \times 10^{-14} \text{ N})(1 \times 10^{-3} \text{ meter}) = 1 \times 10^{-17} \text{ J}$$

Part (b): If we choose the point X as the standard position, then the electron's potential energy at this point is just $U_x = 0$. To find the electron's potential energies at the points Y and Z , use the values of work you have calculated and the relations between these works and the electron's potential energies: $W_{xy} = U_x - U_y$, $W_{xz} = U_x - U_z$, and $W_{yz} = U_y - U_z$. (You only need two of these relations, but the third provides a useful check.)

ANSWERS TO PROBLEMS

1. 20 m/s
2. $v = \sqrt{2k_e qQ/mR}$
3. A
4. a. Child A
b. 0.25 meter
c. 3.7 m/s
101. only (b)
- 102.
- | Standard Position | Potential Energies (joule) | | | |
|-------------------|----------------------------|-----------------|------------------|------------------|
| | U_a | U_b, U_e | U_c | U_d |
| B or E | 2×10^4 | 0 | -1×10^4 | -2×10^4 |
| C | 3×10^4 | 1×10^4 | 0 | -1×10^4 |
103. a. $W_{ab} = U_a - U_b = 2 \text{ J}$, $W_{bc} = U_b - U_c = -4 \text{ J}$
b. $U_d = -3 \text{ J}$ (note the sign)
104. a. $W_{xy} = 1 \times 10^{-17} \text{ J}$, $W_{xz} = 1 \times 10^{-17} \text{ J}$, $W_{yz} = 0$
b. $U_x = 0$, $U_y = -1 \times 10^{-17} \text{ J}$, $U_z = -1 \times 10^{-17} \text{ J}$
c. $U_x = +1 \times 10^{-17} \text{ J}$, $U_y = 0$, $U_z = 0$
105. a. lower
b. largest at top of pass, smallest at both beach and Death Valley
106. a. $U_b = 0$, $U_a = -6.4 \times 10^{-18} \text{ J}$
b. process is different, work is the same
107. a. $E = 0$
b. $E = 0$, $U = -25 \text{ J}$, $K = E - U = 25 \text{ J}$
c. 10 m/s
108. The work done by \vec{F}_f is larger for a longer path between two points, and thus is not independent of process.

109. a.

	W	K	U	E
Kind	number	number	number	number
Signs	+, 0, -	+, 0	+, 0, -	+, 0, -
Unit	joule	joule	joule	joule

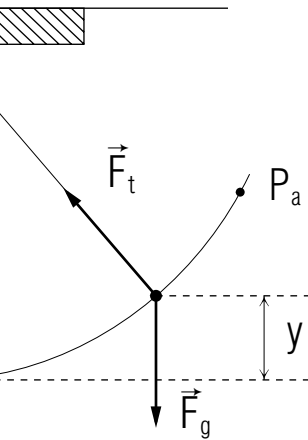
- b. possible signs
c. potential energy and energy
110. a. +, smaller
b. 0, equal
c. -, larger
d. -, larger
111. $1 \times 10^7 \text{ m/s}$
112. a. $E = 1/2mv^2 + mgy$
b. State a: 0.08 J, State b: 0, State c: -0.04 J
113. a. It increases
b. They increase
- 114.
- | State | K | U | E |
|-------|-----------|-----------|-----------|
| a | different | same | different |
| b | same | same | same |
| c | same | different | different |
115. a. $-5.4 \times 10^{33} \text{ J}$ (note the sign)
b. $4 \times 10^{-16} \text{ J}$
c. zero
116. $E = K + U$ is only conserved if U is due to *all* forces doing work. A force other than gravity does work on each particle described.
117. 4.0 meter
118. equation equivalent to: $-\frac{GM}{R_a} = -\frac{GM}{R_b} + \frac{1}{2}v_b^2$
119. a. point B
b. point C
120. a. Largest: C. Smallest: A. Same: B and D
b. Largest: A. Smallest: C.
c. A, C
121. equal to
122. a. $v_e = \sqrt{2GM/R}$

- b. $v_e = 1.1 \times 10^4$ m/s
123. smaller than, equal to
124. a. graph a, larger
b. graph a, larger
c. graph b, smaller
d. graph b, smaller
125. 3×10^{11} J
126. 6×10^7 m/s
127. 10 cm
128. a. either relation
b. only work and kinetic energy
129. a. $V_e = \sqrt{(2GM/R) + (2GM_s/R_0)}$
b. larger
c. $V_e = 4.4 \times 10^4$ m/s
130. a. same
b. yes
c. smaller

MODEL EXAM

1. **Energies of a proton scattered by a nucleus.** As a proton scatters from a fixed nucleus, it follows the path shown in the following drawing under the sole influence of the Coulomb electric force \vec{F}_e exerted by the nucleus.

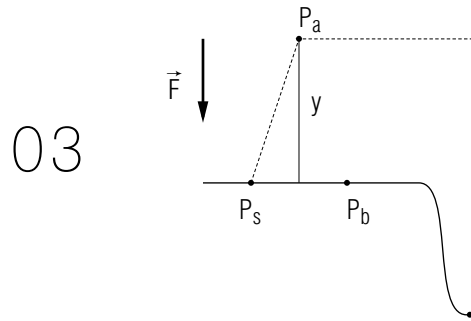
Collated Graphics



00



Fig A-3

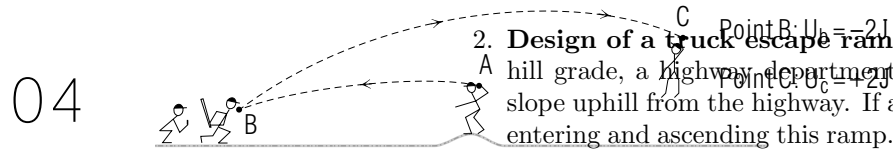


03

Consider the proton's Coulomb potential energy U due to its interaction with the nucleus, and its energy E due to this interaction.

- At which of the indicated points is the proton's potential energy U larger than it is at the point O ? (Your answer may include none or more than one of these points.)
 - At which of the indicated points is the proton's energy E larger than it is as the point O ? (Your answer may include none or more than one of these points.)
- When the proton moves from the point O to a point P not indicated on the drawing, the work done on the proton by the force \vec{F}_e is negative.
- Is the proton's potential energy U at the point O larger than, equal to, or smaller than its potential energy at the point P ?
 - If an identical proton later moves from the point O to the point P along a much shorter path, is the work done on it by the force \vec{F}_e larger than, smaller than, or the same as the work done on the proton we are considering?

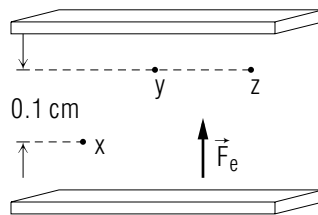
Fig A-4



04

2. Design of a truck escape ramp. At intervals along a steep downhill grade, a highway department builds steep "escape ramps" which slope uphill from the highway. If a truck loses its brakes, it can stop by entering and ascending this ramp. Suppose a truck of mass 1.0×10^4 kg has brake failure, and enters the ramp at the bottom with a speed of 30 m/s. The truck moves with negligible friction up the ramp, and comes to rest just at the top of the ramp.

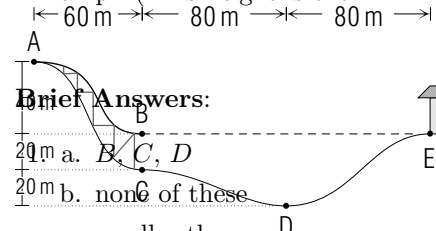
Fig A-5



05,06

Fig A-6

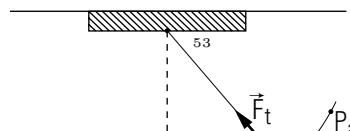
What is the height of the top of the ramp above the bottom of the ramp? (This height is the minimum one should have for safety.)



Brief Answers:

- B, C, D
 - none of these
 - smaller than
 - same as
2. 45 meter

Fig B-1



07

02

