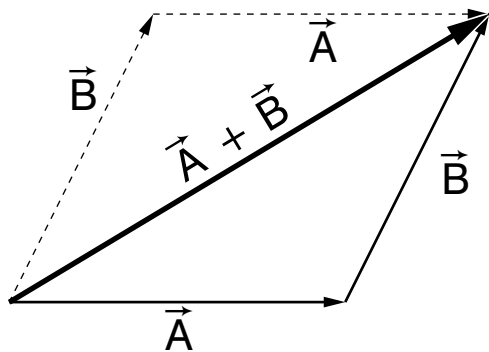


VECTORS



VECTORS

by

F. Reif, G. Brackett and J. Larkin

CONTENTS

- A. Position and Displacement
- B. Vectors and Vector Equality
- C. Vector Multiples and Unit Vectors
- D. Addition of Vectors
- E. Subtraction of Vectors
- F. Vector Equations
- G. Summary
- H. Problems

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Input Skills:

1. Vocabulary: length (MISN-0-403).
2. State the difference between the value and the magnitude of a number (MISN-0-403).

Output Skills (Knowledge):

- K1. Vocabulary: position, speed, position vector, reference frame, coordinate system, displacement, vector, angle between two vectors, unit vector, angstrom (/AA).
- K2. Define the multiplication of a vector by a number and illustrate with a drawing.
- K3. Define the addition and subtraction of two vectors and illustrate with drawings.

Output Skills (Problem Solving):

- S1. Given a quantity, state whether it is a vector or a number, and determine its magnitude.
- S2. Given two vectors, compare their magnitudes and directions and determine whether they are equal.
- S3. Given a vector in one of these forms, represent it in any of the others: (a) algebraic symbol; (b) arrow symbol; (c) a unit vector multiplied by a number.
- S4. Determine the sum or difference of two vectors.
- S5. Determine the product for a given vector multiplied by a number.
- S6. Given a diagram representing the sum or difference of two vectors, write an algebraic equation expressing this relationship.
- S7. Given a vector equation, solve it for any quantity in the equation.

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MISN-0-405

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Abstract:

Since most observed phenomena involve spatial relationships, we need a useful method which can describe such relationships more easily than traditional geometry or trigonometry. The method of “vectors” is designed to achieve this aim by expressing complex spatial relationships systematically in terms of combinations of simple standard relationships.

Vectors are extremely useful in all the sciences and will be used extensively throughout this course. For example, vectors are used to describe the motion of objects through space; to study the spatial arrangement of atoms in molecules (such as proteins or nucleic acids); to analyze the directional properties of light manifested by “polarization” effects (of the type used in polaroid sunglasses or in some methods of chemical analysis); or to deal with some medical applications (such as the “vector electrocardiogram” useful for the diagnosis of heart disease).

SECT.

A POSITION AND DISPLACEMENT

A scientific description of spatial relationships cannot remain on a purely abstract symbolic level, but must ultimately refer to some observable objects. Hence the description of the simplest type of spatial relationship, the definition of position, must involve a comparison with observable objects.

Def.	Position of a point: An unambiguous specification of the relationship of this point relative to some chosen set of observable objects.	(A-1)
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This chosen set of objects is called a “reference frame.”

Def.	Reference frame: The set of objects relative to which the position of any point is specified	(A-2)
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(The objects are chosen so that their spatial relationship remains unchanged in time.) For example, the floors and walls of a laboratory might constitute a suitable reference frame.

The specification of position can be achieved in a simple standardized way by choosing some convenient “coordinate system” fixed relative to the reference frame and defined this way:

Def.	Coordinate system: A point O (called the “origin”) and a set of mutually perpendicular directions specified relative to some reference frame.	(A-3)
------	--	-------

The directions can be indicated by arrows and are denoted by symbols such as \hat{x} , \hat{y} , or \hat{z} . (These directions are often indicated by directed lines, called “axes,” drawn through the origin O .) Fig. A-1 shows several examples of such coordinate systems. To specify the position of a point along a line, a coordinate system with a single direction is sufficient. (See Fig. A-1a and Fig. A-1b.)

To specify the position of a point anywhere in a plane, a coordinate system with two axes is sufficient. (See Fig. A-1c.) To specify the position of a point anywhere in space, a coordinate system with three axes is required. (See Fig. A-1d.) The number of directions (axes) of a coordinate system is called its “dimensionality.”

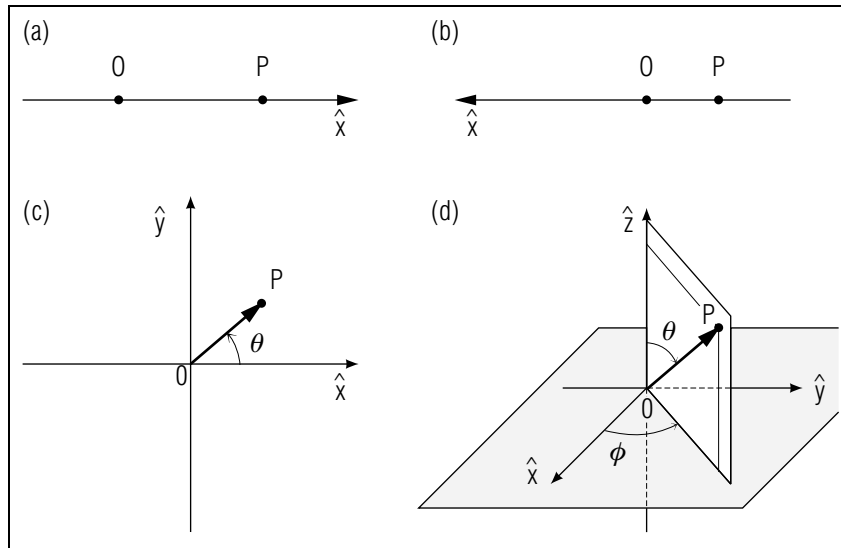


Fig. A-1: Specification of the position of a point P relative to a coordinate system. (a) Coordinate system along a line. (b) Description of the same point P relative to a different coordinate system along this line. (c) Coordinate system in a plane. (d) Perspective drawing of a coordinate system in space.

Note that one is free to choose *any* coordinate system for a given reference-frame. (For example, Fig. A-1a and Fig. A-1b show two different coordinate systems which might be chosen to describe the position of the same point P along a given line.) Hence it is convenient to choose a coordinate system which leads to the simplest possible description of the positions of all points of interest.

Once a coordinate system has been specified, the position of a point P relative to this coordinate system (and associated reference frame) can be specified by providing these two items of information:

- (1) the distance between the origin O and the point P .
- (2) the direction of the straight line from O to P (specified relative to the directions of the coordinate system).

For example, in the case of a one-dimensional coordinate system, the *direction* can be described by specifying whether it is along \hat{x} or opposite to \hat{x} in Fig. A-1a.

In the case of a two-dimensional coordinate system, the *direction* can be described by one angle measured from one axis to the vector in a specified rotational direction. This angle is usually called θ and is so-labeled in Fig. A-1c.¹

In the case of a three-dimensional coordinate system, the *direction* of any vector is described by two angles θ and ϕ , always labeled as such, as shown in Fig. A-1d. Note that here, in three dimensions, the angle θ is always the angle between \hat{z} and the vector (in contrast to the case of two dimensions).

The specification of position by distance and direction is, of course, very familiar from everyday life (e.g., 5 miles northwest from Boston).*

* Position can also be described equivalently by specifying several lengths and signs (“coordinates”) without specifying any angles. We shall discuss this alternative description in Unit 407.

To abbreviate our wording, it is convenient to introduce the word “displacement” defined this way:*

* This definition of “displacement” supersedes our earlier use of this word in the preceding unit.

Def. **Displacement:** A quantity specified jointly by a distance *and* a direction. (A-4)

Then we can say that the position of a point P can be specified by its displacement from the origin O (i.e., by the displacement from O to P). Similarly, the position of a point P relative to any other point Q can be specified by the displacement from Q to P , as indicated in Fig. A-2. (The length of this displacement is the distance between Q and P ; the direction of this displacement can be described by suitable angles measured relative to the directions of some coordinate system.)

¹The angle would still be called θ if the axes in the figure were labeled \hat{x} and \hat{z} , or whatever.

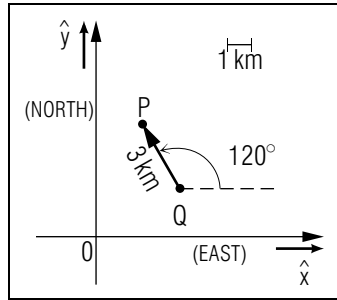


Fig. A-2: Specification of the position of a point P relative to Q by the displacement from Q to P . (The displacement shown is 3 km at an angle of 120° toward the north from the eastern direction \hat{x} .)

A “particle” (i.e., a small object) is said to “move” relative to a reference frame if its position changes relative to this frame. The change of position of a particle moving from a point Q to a point P can then be specified by its displacement from Q to P . (This displacement does *not*, however, describe the *path* by which the particle moves from Q to P since the particle need not necessarily move along a straight line between these points.)

Knowing About Displacements

A-1 Find each of the following displacements, expressing your answers in two ways, using compass directions and using one of the directions \hat{x} , \hat{y} , and \hat{k} indicated in Fig. A-3. (a) Through what displacement does a dog move as he travels from his home to the point A along the path shown in Fig. A-3? (b) Starting at the point A , the dog moves through a displacement of 70 meter opposite to \hat{y} , arriving at the point B where he buries a bone. Through what displacement does the dog move as he returns home from B ? (c) What is the displacement from the dog’s home to the bone? (*Answer: 101*)

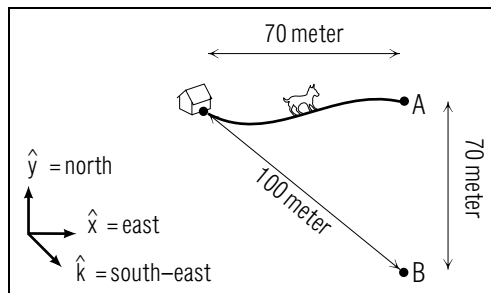


Fig. A-3: Note: \hat{x} , \hat{y} , and \hat{k} are all in the same plane. Also, $\hat{k} = (\hat{x} - \hat{y})/\sqrt{2}$.

SECT.

B VECTORS AND VECTOR EQUALITY

We can now consider a quantity somewhat more general than a displacement by introducing this definition:

Def.	Vector: A quantity specified jointly by a magnitude <i>and</i> a direction (where the “magnitude” is some positive number).	(B-1)
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A displacement is thus a special case of a vector having a magnitude which is a distance (i.e., a length). More generally, however, the magnitude of a vector need *not* be a length. [For example, the magnitude might be a length divided by a time (i.e., a speed) with units of meter/second.] The direction of a vector must be specified in the same way as that of a displacement by suitable angles relative to some coordinate system.

The algebraic symbol used to denote a vector is an alphabetic letter printed in boldface type (such as A) or a letter with an arrow above it (such as \vec{A}).*

We here denote a vector by writing an alphabetic letter with an arrow above it (such as \vec{A}).

The *magnitude* of the vector \vec{A} is denoted by $|\vec{A}|$ or simply by A (i.e., by the letter A without boldface type or any other vector designation).

Note that a vector must always be specified *both* by a magnitude *and* a direction. For example, the velocity \vec{v} of a particle is defined as a vector having a magnitude equal to the speed of the particle and a direction along the direction of motion of the particle. Then we might say that the vector $\vec{v} = 50$ kilometer/hour east, but that its magnitude is $|\vec{v}| = 50$ kilometer/hour (*without* any specification of direction).

Graphically (i.e., pictorially) a vector can be represented by an arrow having the same direction as the vector (relative to the specified coordinate system) and having a length proportional to the magnitude of the vector. The place where the vector is drawn is irrelevant. (See the arrows representing the vector \vec{A} in Fig. B-1.) The scaling factor relating the length of the arrow to the magnitude of the vector can be chosen in any convenient way, but must be kept the same in a given discussion. (For example, if the vector \vec{A} in Fig. B-1 is 2 km/hour east, a vector \vec{D} which

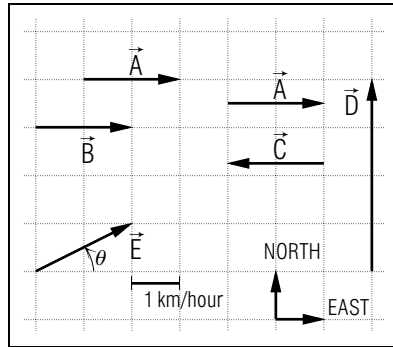


Fig. B-1: Graphical representation of vectors by arrows. All the vectors shown have the same unit (km/hour).

is 4 km/hour north must be represented by an arrow which has twice the length of the arrow representing the vector \vec{A} .)

The “vector zero” is a vector having a magnitude equal to zero. This vector requires no specification of direction. It is represented graphically by a dot and is denoted by the symbol $\vec{0}$ (or simply by 0).

REMARK

It is easy to find the magnitude and direction of vectors represented by arrows drawn on a grid. For example, in Fig. B-1 the magnitude E of the vector \vec{E} is, by the Pythagorean theorem, such that $E^2 = (2\text{ km/hr})^2 + (1\text{ km/hr})^2 = 5(\text{ km/hr})^2$, so that $E = 2.2\text{ km/hr}$. Furthermore, the angle θ between the direction of E and the east direction is such that $\tan \theta = 1/2 = 0.5$, so that $\theta = 27^\circ$. The definition of a vector, Def. (B-1), suggests this definition of the equality of two vectors:

$$\text{Def. } \left| \begin{array}{l} \mathbf{Equality\ of\ vectors:} \text{ Two vectors are equal if,} \\ \text{and only if, they have both the same magnitude} \\ \text{and the same direction.} \end{array} \right. \quad (\text{B-2})$$

The equality of two vectors \vec{A} and \vec{B} is expressed symbolically by writing:

$$\vec{A} = \vec{B} \quad (\text{B-3})$$

Note that the arrows representing equal vectors may be located anywhere as long as they have the same length and direction. (See, for example, the vectors \vec{A} and \vec{B} in Fig. B-1.) On the other hand, the vectors \vec{A} and \vec{C} in

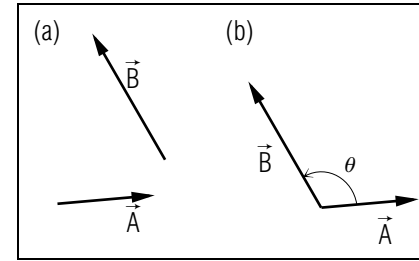


Fig. B-2: Angle θ between two vectors. (a) Arrows drawn at arbitrary places. (b) Arrows drawn from the same point. (The angle θ between \vec{A} and \vec{B} is 120° in the diagram.)

Fig. B-1 are not equal (despite the equality $A = C$ of their magnitudes) because their directions are different.

The angle between two vectors is defined in this way, illustrated in Fig. B-2:

$$\text{Def. } \left| \begin{array}{l} \mathbf{Angle\ between\ two\ vectors:} \text{ The angle (be-} \\ \text{tween } 0^\circ \text{ and } 180^\circ) \text{ between the arrows represent-} \\ \text{ing the vectors when these arrows are drawn from} \\ \text{the same point.} \end{array} \right. \quad (\text{B-4})$$

The value of a vector must be specified by information about its magnitude and its direction relative to some coordinate system. Thus a vector must be carefully distinguished from a simpler kind of quantity which can be completely specified by a single number (with attached sign) and which is called a “scalar.” For example, a temperature (such as -14 degree) is a scalar.

POSITION VECTOR

The directions of a coordinate system can be specified by vectors \hat{x} , \hat{y} , and \hat{z} along these directions. (The magnitudes of these vectors are conventionally chosen equal to 1.)

As discussed in Sec. A, the position of a point P relative to some coordinate system can be specified by its displacement \vec{r} from the origin O of this coordinate system, as illustrated in Fig. B-3. This displacement \vec{r} is called the “position vector” of P in accordance with this definition:

$$\text{Def. } \left| \begin{array}{l} \mathbf{Position\ vector:} \text{ The position vector } \vec{r} \text{ of a point} \\ P \text{ relative to a coordinate system with origin } O \text{ is} \\ \text{the displacement from } O \text{ to } P. \end{array} \right. \quad (\text{B-5})$$

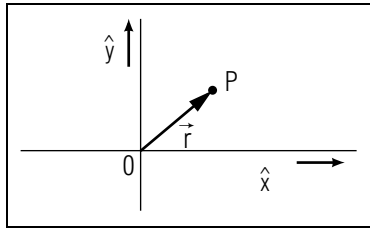


Fig. B-3: Position vector \vec{r} of a point P relative to a two-dimensional coordinate system.

Vectors and Their Magnitudes

B-1 A man walks with a velocity \vec{V} , which has the unit km/hour. (a) What two algebraic symbols are commonly used for the magnitude of a vector \vec{V} ? (b) What is the unit of this magnitude? (c) Can this magnitude be positive? negative? zero? (d) If $\vec{V} = 4$ km/hour north, what is the magnitude of \vec{V} ? (Answer: 107)

Arrow Symbols for Vectors (Cap. 2)

B-2 (a) Using a grid like the one in Fig. B-4, draw an arrow symbol representing the vector $\vec{A} = 6$ meter in the direction opposite to \hat{y} . (b) Describe the vector \vec{B} in Fig. B-4 by finding its magnitude B , and the angle θ between B and the direction \hat{x} . (Answer: 103) (Suggestion: [s-9])

B-3 (a) What is the value of each of the four vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3,$ and \vec{a}_4 shown in Fig. B-5? (b) Which two of these vectors are equal? (c) Specify the direction of \vec{a}_3 by choosing one of the four directions labeled by an angle from \hat{x} . (Answer: 105)

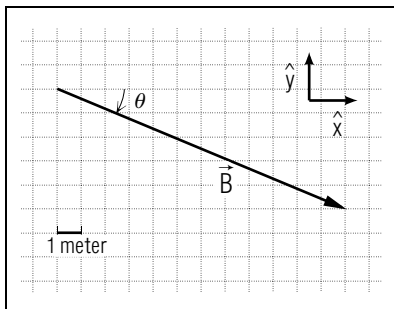


Fig. B-4.

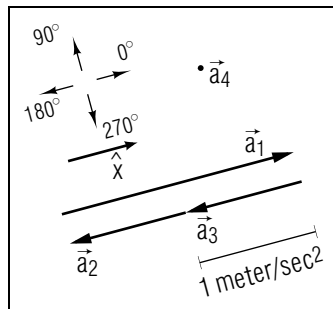


Fig. B-5.

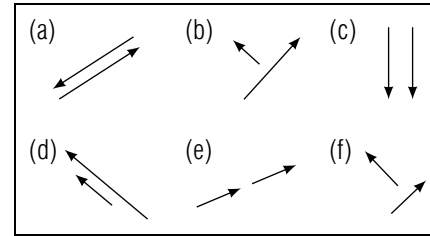


Fig. B-6.

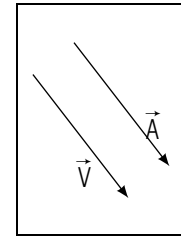


Fig. B-7.

Comparing Vectors and Numbers (Cap. 1a)

B-4 State whether each of these quantities is a vector or a number: (a) the temperature -12 (degree Fahrenheit); (b) the displacement from a ground control station to an airplane 5000 meter directly above; (c) the velocity 20 meter/sec west; (d) the distance 2 meter between two windows. What is the sign of each number? What is the direction of each vector? What is the magnitude of each quantity? (Answer: 102) (Suggestion: [s-2])

Vectors, Magnitudes, Directions (Cap. 1b)

B-5 In which of the six pairs of vectors in Fig. B-6 are the magnitudes of the two vectors equal? In which pairs are the directions of the two vectors equal? In which pairs are the two vectors equal? (Answer: 104)

B-6 The vectors \vec{A} and \vec{V} in Fig. B-7 have magnitudes $A = 5$ meter and $V = 5$ meter/sec Are these vectors equal? (Answer: 106)

SECT.

C VECTOR MULTIPLES AND UNIT VECTORS

Any vector which is parallel to some vector \vec{A} can be easily related to \vec{A} . This can be done by defining a multiple of a vector, i.e., the vector resulting from multiplying a vector \vec{A} by an ordinary *number* m (such as $m = 3$ or $m = -0.25$ second). Note that the number m may include attached signs and units.

Def.	<p>Multiple of a vector: The vector $m\vec{A}$ (or $\vec{A}m$) is the vector which has a magnitude m times as large as the magnitude of \vec{A}. Its direction is the same as that of \vec{A} if m is positive, and is opposite to \vec{A} if m is negative.</p>	(C-1)
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(The magnitude of the vector $m\vec{A}$ is thus simply $|m||A|$.) Division of a vector by a number n is defined to be the same as multiplication of this vector by $(1/n)$. Fig. C-1 shows some examples.

The vector $-\vec{A}$ corresponds to the special case where the number $m = -1$. Thus the vector $-\vec{A}$ has the same magnitude as \vec{A} , but the opposite direction. (See Fig. C-1.)

Note that we have *not* defined how to multiply a vector by another vector.

A direction alone, irrespective of any magnitude, can be specified most conveniently by a vector having this direction and a magnitude simply equal to 1. Such a vector is called a “unit vector” in accordance with this definition:*

* The word “unit” is here used to denote “unity” (i.e., the number 1) rather than a unit symbol describing some standard of measurement.

Def.	<p>Unit Vector: A vector having a magnitude equal to the number 1 (without any units). When a unit vector is plotted on a graph, only its direction has meaning.</p>	(C-2)
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For example, the mutually perpendicular directions of a coordinate system are customarily specified by *unit* vectors denoted by \hat{x} , \hat{y} , and \hat{z} .

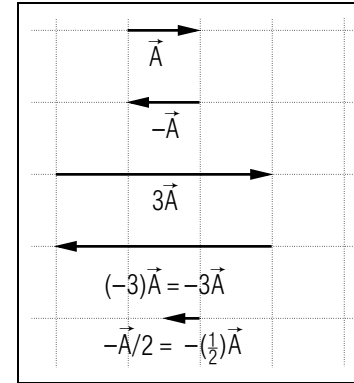


Fig. C-1: Some multiples of the vector \vec{A} .

VECTORS PARALLEL TO A SPECIFIED DIRECTION

Suppose that some direction is specified by a unit vector \hat{x} , as indicated in Fig. C-2. Then any vector parallel to \hat{x} can be expressed as a multiple of \hat{x} . For example, a vector \vec{B} having a magnitude of 2 meter and a direction *along* \hat{x} can be written as $\vec{B} = (2 \text{ meter})\hat{x}$. Similarly, a vector \vec{C} with a magnitude of 3 meter/second opposite to \hat{x} can be written as $\vec{C} = (-3 \text{ meter/second})\hat{x}$.

Quite generally, any vector \vec{B} parallel to \hat{x} can thus be simply written as:

$$\vec{B} = b\hat{x} \tag{C-3}$$

where b is some *number* called the “numerical component” of \vec{B} along \hat{x} . Indeed, the *magnitude* of the vector \vec{B} is then equal to the magnitude of the number b . Furthermore, the direction of \vec{B} is specified by the sign of the number b (since this direction is along \hat{x} if b is positive, and is opposite to \hat{x} if b is negative.)

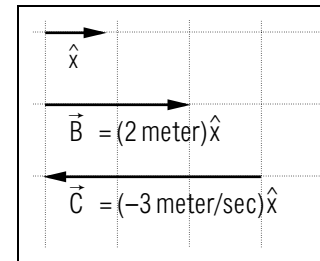


Fig. C-2: Vectors parallel to a given direction specified by the unit vector \hat{x} . [See Def. (C-2) above.]

Now: Go to tutorial section C.

Vectors Times Numbers (Cap. 3)

C-1 Draw any arrow representing a vector \vec{X} . Then draw arrows representing $3\vec{X}$, $-\vec{X}$, and $(-2)\vec{X}$. (Answer: 108)

C-2 A car rolling downhill after a failure of its brakes reaches, after a time t , a velocity $\vec{v} = \vec{a}t$ where $\vec{a} = 1.5$ meter/sec² along the unit vector \hat{x} pointing downhill along the road. (a) What is the velocity \vec{v} when $t = 2.0$ sec? (b) Just after hitting a brick wall, the car has a velocity \vec{v}' equal to (-0.10) times its velocity when $t = 2.0$ sec. What is \vec{v}' ? (Answer: 110)

Interchanging Vector Representations (Cap. 2)

C-3 (a) On a grid like the one in Fig. C-3 draw arrows representing the vectors $\vec{A} = (-4 \text{ km})\hat{x}$ and $\vec{B} = -1/2\vec{A}$. (b) Express the vector \vec{C} shown in Fig. C-3 as a multiple of one of the unit vectors \hat{x} and \hat{y} . (Answer: 112)

C-4 A urea molecule consists of carbon, hydrogen, oxygen, and nitrogen atoms all lying in a plane as shown in Fig. C-4. The symbol \AA stands for angstrom $\equiv 10^{-10}$ meter. (a) What are the displacements \vec{X} , \vec{Y} , and \vec{Z} from the nitrogen atom N_a to the adjacent atoms H_a , H_b , and C ? (b) Express the displacement \vec{X} as a multiple of the unit vector \hat{x} . (Answer: 114) ([s-7], [p-1])

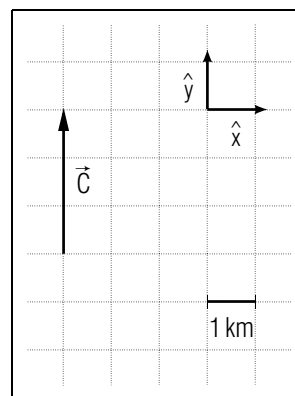


Fig. C-3.

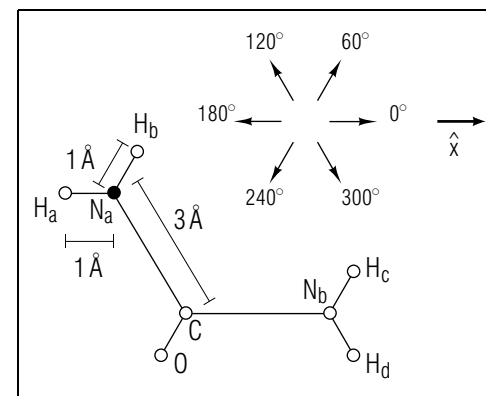


Fig. C-4.

SECT.

D ADDITION OF VECTORS

Suppose that a boat moves by a displacement \vec{A} from a point P_1 to a point P_2 , and then moves by a displacement \vec{B} from P_2 to some point P_3 . (See Fig. D-1.) Then the total displacement of the boat is the displacement \vec{S} from P_1 to P_3 . In other words, the combination of two successive displacements \vec{A} and \vec{B} is equivalent to a third displacement \vec{S} .

By analogy, we can define a general procedure (called “vector addition”) whereby any two vectors \vec{A} and \vec{B} can be combined to yield a third vector \vec{S} which is called their “vector sum” and which is denoted by:

$$\vec{S} = \vec{A} + \vec{B} \quad (\text{D-1})$$

This addition process is defined by the following procedure, illustrated in Fig. D-2:

Def. Vector addition: To add \vec{B} to \vec{A} , draw arrows representing these vectors so that the tip of \vec{A} coincides with the tail of \vec{B} . Then the vector sum ($\vec{A} + \vec{B}$) is represented by the arrow drawn from the tail of \vec{A} to the tip of \vec{B} .	(D-2)
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It is important to note that we have given new meanings to the word “sum” and to the “plus” sign used to relate the vectors \vec{A} and \vec{B} in Eq. (D-1). These meanings are distinct from those attributed to the word “sum” and to the “plus” sign when these are used in connection with ordinary numbers.

Since, in Def. (D-2), the addition process has been defined without mention of any specific coordinate system, the statement that “the vector \vec{S} is the sum of two other vectors \vec{A} and \vec{B} ” is true irrespective of the choice of coordinate system used to specify the directions of these vectors.

The “zero vector” has the expected property that $\vec{A} + \vec{0} = \vec{A}$.

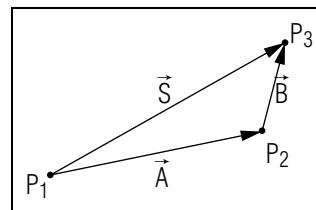


Fig. D-1: Successive displacements from P_1 to P_2 and from P_2 to P_3 .

PROPERTIES OF VECTOR ADDITION

What happens when two vectors are added in opposite order? In the parallelogram of Fig. D-3, the diagonal represents the vector \vec{S} which is the sum ($\vec{A} + \vec{B}$) obtained when the vector \vec{B} is added to the vector \vec{A} (as shown by the solid arrows). But \vec{S} is also the sum ($\vec{B} + \vec{A}$) obtained when the vector \vec{A} is added to the vector \vec{B} (as shown by the dashed arrows). Thus we see that:

$$\vec{B} + \vec{A} = \vec{A} + \vec{B} \quad (\text{D-3})$$

In other words, vectors can be added in any order without affecting the result.

What happens when several vectors are added in different combinations? In Fig. D-4 the vector \vec{S} is seen to be the sum $(\vec{A} + \vec{B}) + \vec{C}$ obtained when the vectors \vec{A} and \vec{B} are first added, and the vector \vec{C} is then added to their sum ($\vec{A} + \vec{B}$). But the vector \vec{S} is also seen to be the sum $\vec{A} + (\vec{B} + \vec{C})$ obtained when \vec{B} and \vec{C} are first added, and their sum ($\vec{B} + \vec{C}$) is then added to \vec{A} . Thus it is apparent that:

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}) \quad (\text{D-4})$$

In other words, vectors can be added in any combination without affecting the result. The sum $\vec{A} + \vec{B} + \vec{C}$ can thus be written without ambiguity

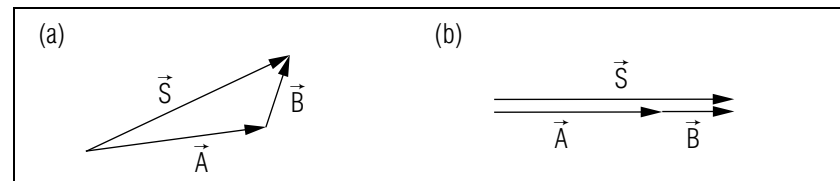


Fig. D-2: Addition of two vectors \vec{A} and \vec{B} to find their sum $\vec{S} = \vec{A} + \vec{B}$. (a) General case. (b) Special case where the vectors have the same direction.

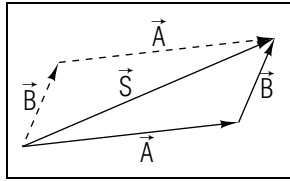


Fig. D-3: Sum of two vectors added in the order $\vec{A} + \vec{B}$ (solid arrows) or in the order $\vec{B} + \vec{A}$ (dashed arrows).

by omitting all parentheses (since it does not matter *how* the vectors are added).

What happens to a sum of vectors when all vectors are multiplied by the same number? The triangle formed by the arrows in Fig. D-5a shows the vector sum $\vec{A} + \vec{B} = \vec{S}$ produced by the addition of the vectors \vec{A} and \vec{B} .

Figure D-5b shows the similar triangle which results when all these vectors are multiplied by the same number m . According to this diagram, $m\vec{A} + m\vec{B} = m\vec{S}$. Thus:

$$m\vec{A} + m\vec{B} = m(\vec{A} + \vec{B}) \quad (\text{D-5})$$

Hence it does not matter whether vectors are multiplied by the same number m *before* being added, or whether they are multiplied by this number *after* being added.

The relations (D-3), (D-4) and (D-5) are exactly the same as those which would be true if the vector symbols denoted ordinary numbers. Hence our entire preceding discussion can be summarized by this state-

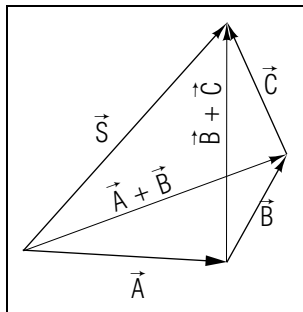


Fig. D-4: Sum \vec{S} of three vectors \vec{A} , \vec{B} , and \vec{C} added in different combinations.

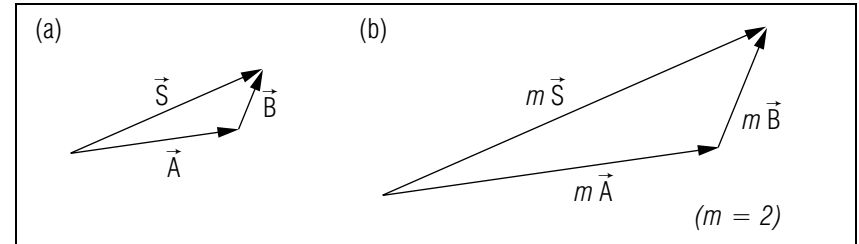


Fig. D-5: Relation between vector addition and multiplication by the same number. (a) Arrows representing the vectors \vec{A} and \vec{B} and their sum $\vec{S} = \vec{A} + \vec{B}$. (b) Multiplication of the preceding vectors by the same number $m = 2$.

ment:

Algebraic symbols representing vectors can be manipulated like ordinary numbers in all processes involving the addition of such symbols or their multiplication (or division) by numbers. (D-6)

Although the manipulation of vector *symbols* is the same as if these denoted ordinary numbers, the *interpretation* of the processes of addition or multiplication is quite different. (Note also that we have not defined the multiplication or division of a vector by another *vector*.)

Adding Vectors (Cap. 3)

D-1 Using a grid like the one in Fig. D-6, you can construct the sum of the vectors \vec{A} and \vec{B} in two ways. (a) Add \vec{B} to \vec{A} by constructing the sum $\vec{S} = \vec{A} + \vec{B}$. (b) Add \vec{A} to \vec{B} , constructing $\vec{S}' = \vec{B} + \vec{A}$. (c) Use one of the unit vectors \hat{x} and \hat{y} to state the values of \vec{S} and \vec{S}' . The correct sum vectors \vec{S} and \vec{S}' are equal, since two vectors can be added in either order. (*Answer: 116*) (*[s-11], [p-2]*)

D-2 Compare the vectors \vec{A} and \vec{B} in Fig. D-6 with their sum \vec{S} . Is the magnitude of a vector sum always larger than the magnitudes of the vectors added to form this sum? (*Answer: 109*)

D-3 A woman doing errands travels through a displacement \vec{X} from her home to the bank. Then starting at the bank, she travels through a displacement \vec{Y} to a store. For each of the possible values of \vec{X}

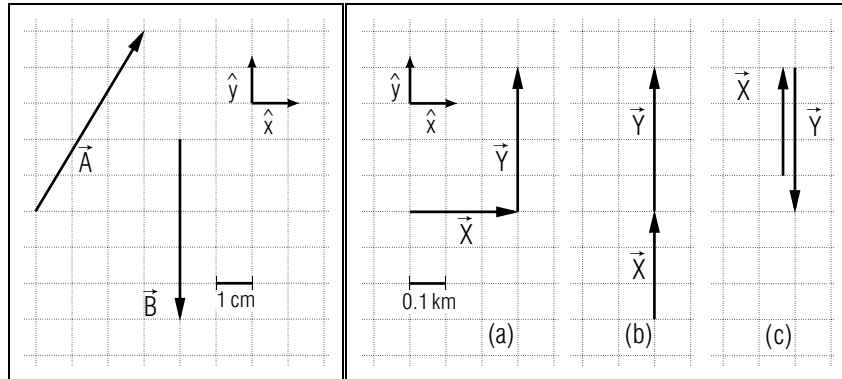


Fig. D-6.

Fig. D-7.

and \vec{Y} shown in Fig. D-7, find the vector sum $\vec{S} = \vec{X} + \vec{Y}$ (the displacement from the woman's home to the store), the magnitude $|\vec{S}| = |\vec{X} + \vec{Y}|$ (the distance from the woman's home to the store), and the numerical sum $X + Y$ (the distance the woman walked). (Answer: 111) ([s-12], [p-3])

D-4 Locations of atoms in a crystal are often specified by their position vectors, i.e., their displacements from some specified atom chosen as the origin O . In a crystal of osmium (the densest metal) one atom has a position vector $\vec{r} = (-3\text{\AA})\hat{x} + (4\text{\AA})\hat{y}$, relative to an atom at O . ($\text{\AA} = \text{angstrom} = 10^{-10}$ meter) (a) Use a grid like the one shown in Fig. D-8 to construct an arrow-representing \vec{r} . (b) What is the distance between the two atoms? (c) Approximately specify the direction of \vec{r} by choosing one of the six directions labeled by a direction from \hat{x} . (Answer: 113) (Suggestion: p-4)

D-5 The three vectors shown in Fig. D-9 can be added in a variety of ways. (a) Find their sum \vec{S} by first constructing $\vec{S}_1 = \vec{X} + \vec{Y}$, and then adding \vec{Z} to \vec{S}_1 to find $\vec{S} = \vec{S}_1 + \vec{Z}$. (b) Find the same sum \vec{S} by the easier method of first finding the sum $\vec{S}_2 = \vec{X} + \vec{Z}$, and then $\vec{S} = \vec{S}_2 + \vec{Y}$. (Answer: 115)

Graphic to Algebraic Symbols (Cap. 4)

D-6 For each of the vector diagrams in Fig. D-10, write an algebraic equation expressing one of the three vectors as a sum of the other two. (Answer: 117) ([s-4], [p-5])

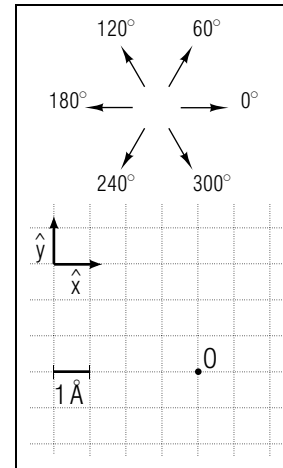


Fig. D-8.

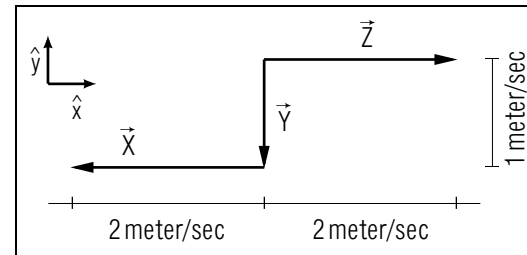


Fig. D-9.

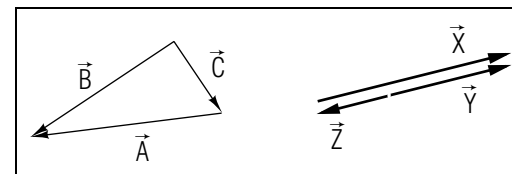


Fig. D-10.

SECT.

E SUBTRACTION OF VECTORS

The subtraction of ordinary numbers is defined as the process inverse to addition. Thus the difference (8 - 5) is defined to be that number which must be added to the subtracted number 5 so as to yield the first number 8. (In other words, $8 - 5 = 3$ because $5 + 3 = 8$.) We shall define the subtraction of a vector \vec{B} from a vector \vec{A} in a precisely analogous way and shall denote the resulting “vector difference” by $\vec{A} - \vec{B}$. Our definition of this subtraction process (and of the associated “minus” sign) is thus:

Def. $\left| \begin{array}{l} \text{Subtraction of vectors: The vector difference} \\ \vec{A} - \vec{B} \text{ is the vector } \vec{D} \text{ which is such that } \vec{B} + \vec{D} = \vec{A} \end{array} \right| \quad (\text{E-1})$

This definition is illustrated in Fig. E-1 and implies this simple rule for subtracting two vectors:

Rule for vector subtraction: To subtract \vec{B} from \vec{A} , draw arrows representing these vectors so that their tails coincide. Then the vector difference ($\vec{A} - \vec{B}$) is represented by the arrow drawn from the tip of \vec{B} to the tip of \vec{A} .

(E-2)

Our Def. (E-1) implies, as expected, that $\vec{A} - \vec{A} = 0$ since $\vec{A} + 0 = \vec{A}$.

Example E-1: Relation between displacement and positions

Suppose that a ship moves from some point P_1 specified by the position vector \vec{r}_1 to some other point P_2 specified by the position vector \vec{r}_2 . Then Fig. E-2 shows that the displacement \vec{D} of the ship is equal to $\vec{D} = \vec{r}_2 - \vec{r}_1$. In other words, the displacement \vec{D} is simply equal to the vector difference of the position vectors.

As illustrated by the parallelogram in Fig. E-3,

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \quad (\text{E-4})$$

i.e., the subtraction of a vector \vec{B} is equivalent to the addition of the vector $-\vec{B}$ which is opposite to \vec{B} . Since subtraction is equivalent to an addition process, Rule (D-6) is also applicable to subtraction. Thus we

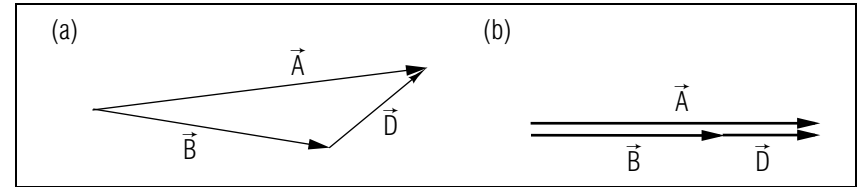


Fig. E-1: Subtraction of two vectors \vec{A} and \vec{B} to find their difference $\vec{D} = \vec{A} - \vec{B}$. (a) General case. (b) Special case where the vectors have the same direction.

arrive at this general conclusion:

Algebraic *symbols* representing vectors can be manipulated like ordinary numbers in all processes involving the addition or subtraction of such symbols, or their multiplication (or division) by numbers.

(E-5)

Note again that, although the manipulation of vector *symbols* is the same as if these denoted ordinary numbers, the *interpretation* of processes represented by the manipulations is different.

Subtracting Vectors (Cap. 3)

E-1 A car travels with a constant speed of 5.2 meter/sec along the curved section of road shown in Fig. E-4. The arrows on the grid indicate the car's velocity \vec{v}_A and \vec{v}_B at the points A and B . To find the rate of change of the car's velocity, one first needs the change $\Delta\vec{v} = \vec{v}_B - \vec{v}_A$. What is this change $\Delta\vec{v}$? What is the change $v_B - v_A$ in the car's speed? (*Answer: 119*) (*Suggestion: [s-6]*)

E-2 Illustrate an alternative method of subtracting vectors by first drawing an arrow representing the vector $-\vec{v}_A$. Then add $(-\vec{v}_A)$

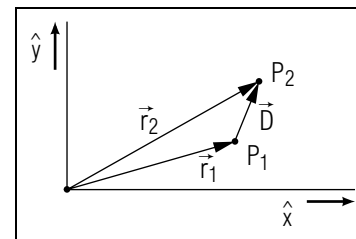


Fig. E-2: Displacement \vec{D} of a ship between two points specified by position vectors \vec{r}_1 and \vec{r}_2 relative to a lighthouse at O .

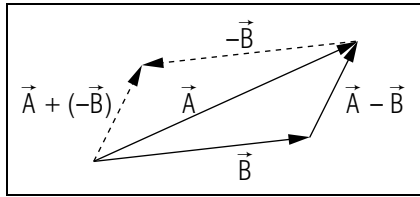


Fig. E-3: Equivalence between the subtraction $\vec{A} - \vec{B}$ [indicated in the lower part of the figure] and the addition $\vec{A} + (-\vec{B})$ [indicated in the upper part of the figure].

to \vec{v}_B to find the difference $\Delta\vec{v} = \vec{v}_B - \vec{v}_A = \vec{v}_B + (-\vec{v}_A)$. Is your value of $\Delta\vec{v}$ the same as the value obtained in problem E-1? (Answer: 121)

E-3 A couple plan a vacation combining climbing a mountain with a stay at a lake, which have position vectors \vec{M} and \vec{L} relative to the couple's home. For each of the possible values of \vec{L} and \vec{M} shown in Fig. E-5, find the vector difference $\vec{D} = \vec{L} - \vec{M}$ (the displacement from the mountain to the lake), the magnitude $|\vec{D}| = |\vec{L} - \vec{M}|$ (the distance between the mountain and the lake), and the numerical difference $L - M$ (the difference in the distances from the couple's home to the two vacation spots). (Answer: 123) ([s-8], [p-6])

E-4 The position vector \vec{B} of Boston from New York is about 300 km northeast, and the position vector \vec{W} of Washington, D.C. from New York is about 330 km southwest. Subtract these vectors to find the displacement $\vec{D} = \vec{W} - \vec{B}$ from Boston to Washington. If an Amtrak train travels through this displacement \vec{D} in a time interval $T = 9$ hour, use the approximate relation $\vec{V} = \vec{D}/T$ to estimate the velocity \vec{V} of the train. (Answer: 125) ([s-3], [p-7])

Graphic to Algebraic Symbols (Cap. 6)

E-5 For each of the vector diagrams shown in Fig. E-6, write an algebraic equation expressing the indicated vector as the difference of

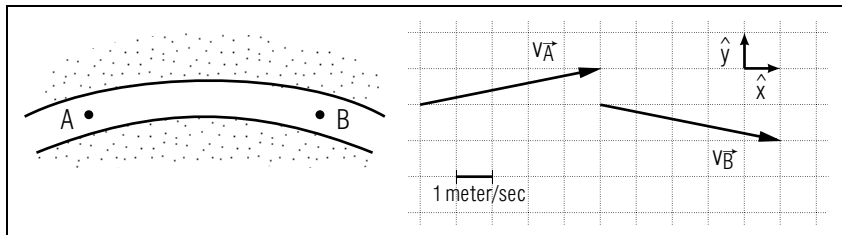


Fig. E-4.

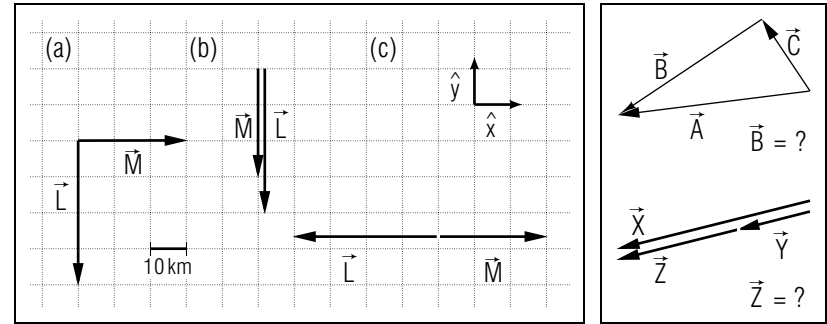


Fig. E-5.

Fig. E-6.

the other two. (Answer: 120) ([s-5], [p-8])

SECT.

F

 VECTOR EQUATIONS

Because of Rule (E-5), any equation relating vectors can be manipulated and solved in a manner similar to that used with equations relating ordinary numbers. For example, if the same vector is added or subtracted to both sides of a vector equation, or if the same *number* is used to multiply or divide both sides of this equation, both sides of the resulting vector equation remain equal. (However, both sides of a vector equation can *not* be multiplied or divided by the same *vector* since multiplication or division by a vector has not been defined.)

Example F-1:

Suppose the vectors \vec{A} and \vec{B} are related so that:

$$\vec{B} - \vec{A} = \frac{1}{2}(\vec{A} + \vec{B})$$

Then we can solve this equation for \vec{B} by these steps:

$$2\vec{B} - 2\vec{A} = \vec{A} + \vec{B}$$

$$\vec{B} = 3\vec{A}$$

Consider an equation such as:

$$a\hat{x} = b\hat{x} \quad (\text{F-1})$$

where a and b are *numbers* and \hat{x} is some unit vector. Although we cannot divide both sides of this equation by the vector \hat{x} , we may still conclude that:

$$a = b \quad (\text{F-2})$$

The reason is that the numbers a and b must have the same *magnitude* to guarantee that the vectors $a\hat{x}$ and $b\hat{x}$ have the same magnitude; furthermore, a and b must have the same *sign* to guarantee that these vectors have the same direction.

The great simplicity achieved by working with vectors is now apparent. Most spatial relationships can be readily expressed as equations

relating combinations of vectors (e.g., sums or multiples of various vectors). But then it is possible to draw conclusions about complex spatial relationships by merely manipulating vector symbols according to the familiar rules of algebra, i.e., *without* needing to use elaborate geometrical reasoning or to visualize complicated spatial relationships in three dimensions.

Solving Vector Equations (Cap. 5)

F-1 Use the equation $\vec{C} = \vec{A} + 2\vec{B}$ and the values $\vec{A} = (5 \text{ meter/sec}^2)\hat{x}$ and $\vec{B} = (-3 \text{ meter/sec}^2)\hat{x}$ to find the vector \vec{C} . Is $C = A + 2B$ a correct relation between magnitudes? (*Answer: 122*)

F-2 To ensure a satisfying splat, a boy leans out a window and tosses a water balloon with a vertically upward velocity \vec{v}_A . As we shall see in the next unit, the motion of the balloon is described by $\vec{v}_B - \vec{v}_A = \vec{a}t$, where \vec{v}_B is the final velocity of the balloon when striking the pavement after a time t in the air. The vector \vec{a} has the constant value $\vec{a} = (-10 \text{ meter/sec}^2)\hat{y}$, where \hat{y} is an upward unit vector. (a) Write an algebraic expression for the velocity \vec{v}_B in terms of the other quantities in the equation provided. (b) Use the values $\vec{v}_A = (5 \text{ meter/sec})\hat{y}$ and $t = 2.0 \text{ sec}$ to find the velocity \vec{v}_B of the balloon at impact. (*Answer: 118*) (*Suggestion: [s-1]*)

SECT.

G

 SUMMARY
DEFINITIONS

position; Def. (A-1)

reference frame; Def. (A-2)

coordinate system; Def. (A-3)

displacement; Def. (A-4)

vector; Def. (B-1)

equality of vectors; Def. (B-2)

angle between two vectors; Def. (B-4)

position vector; Def. (B-5)

multiple of a vector; Def. (C-1)

unit vector; Def. (C-2)

addition of vectors (or vector sum); Def. (D-2)

subtraction of vectors (or vector difference); Def. (E-1)

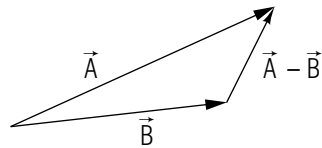
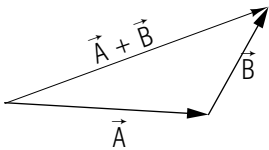
IMPORTANT RESULTS

Multiplication of a vector by a number: Def. (C-1)

$$|m\vec{A}| = |m| \cdot |\vec{A}|$$

$m\vec{A}$ has same direction as \vec{A} if m is positive, opposite direction if m is negative.

Addition and subtraction of vectors: Def. (D-2), Def. (E-1), Rule (E-2)



Manipulation of vector symbols: Rule (E-5)

Algebraic symbols representing vectors can be manipulated like ordinary numbers in all processes involving the addition or subtraction of such symbols, or their multiplication or division by numbers.

NEW CAPABILITIES

You should have acquired the ability to:

- (1) (a) Distinguish vectors from numbers and find their magnitudes. (b) Compare the magnitudes and directions of two vectors, and determine whether they are equal. (Sec. B)
- (2) Use interchangeably these vector representations: (a) algebraic symbol, (b) arrow symbol, (c) specification of magnitude and direction, (d) a number multiplied by a unit vector. (Sects. B and C, [p-1])
- (3) Add or subtract two vectors, or multiply a vector by a number. (Sects. C, D, and E, [p-2], [p-3], [p-4], [p-6], [p-7])
- (4) Using a diagram showing the sum or difference of two vectors, write an algebraic equation expressing this relationship. (Sects. D and E, [p-5], [p-8])
- (5) Manipulate and solve vector equations. (Sec. F)
- (6) Habitually state vectors in correct form. (Sects. C, D, and E)

SECT.

H

 PROBLEMS

Adding Perpendicular Vectors In 3d

H-1 The magnitude of the sum of *two* perpendicular vectors can be found using the Pythagorean theorem. This problem illustrates a very similar relation expressing the magnitude of the sum of *three* mutually perpendicular vectors, such as those shown in Fig. H-1. The vector \vec{S} , the displacement from the corner P_1 of a rectangular block to the diagonally opposite corner P_2 , is the sum of the three perpendicular vectors \vec{A} , \vec{B} , and \vec{C} . Suppose \vec{A} , \vec{B} , and \vec{C} have the magnitudes $A = 3$ meter, $B = 4$ meter, and $C = 12$ meter. (a) Apply the Pythagorean theorem to the perpendicular vectors \vec{A} and \vec{B} , to find the magnitude S of their sum, and an expression for S_1^2 in terms of the magnitudes A and B . (b) Apply the Pythagorean theorem to the right triangle including the vectors \vec{S}_1 , \vec{C} , and \vec{S} in order to find the magnitude S , and an expression for S^2 in terms of S_1 and C . (c) Write a general expression for S^2 in terms of A , B , and C . (*Answer: 124*)

H-2 Many crystal properties depend on the distance between nearby atoms. For example, consider a part of a cesium chloride crystal (shown in Fig. H-2) which consists of 8 Cs atoms at the corners of a cube surrounding one chloride (Cl) atom at its center. This cube is approximately 4.0 \AA in length. Use the procedure described in problem H-1 to find the distance between the Cs atom labeled O and its nearest-neighbor Cl atom at the center of the cube. (*Answer: 126*)

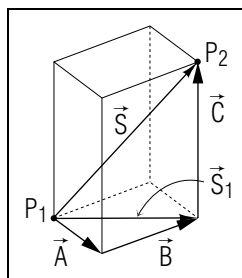


Fig. H-1.

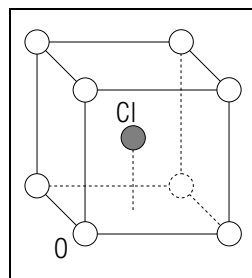
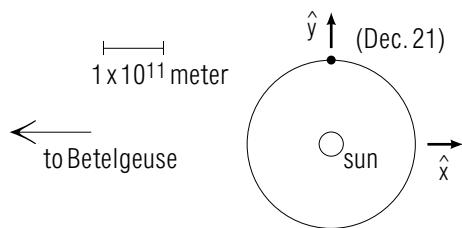


Fig. H-2.

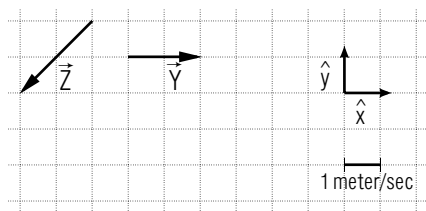
PRACTICE PROBLEMS

p-1 *INTERCHANGING VECTOR REPRESENTATIONS (CAP. 2):* The earth travels in a circular path at a constant distance of 1.5×10^{11} meter from the sun. On December 21, the earth has the position shown in this drawing: ▶



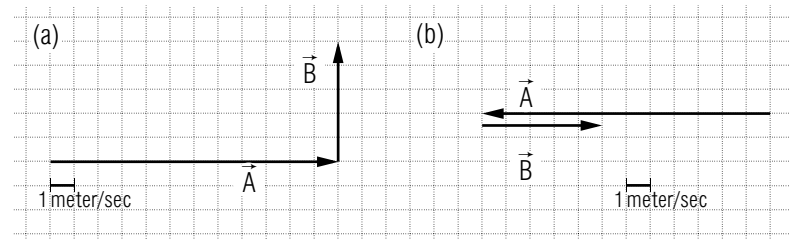
(a) Draw an arrow representing the displacement \vec{R}_1 from the sun to the earth on December 21. (b) Express \vec{R}_1 using one of the unit vectors \hat{x} and \hat{y} . (c) Three months later, on March 21, the position vector \vec{R}_2 of the earth relative to the sun has the direction from the sun towards the star Betelgeuse. On the preceding diagram, draw an arrow beginning at the sun and representing \vec{R}_2 . Then express \vec{R}_2 using one of the unit vectors. (Answer: 4) (Suggestion: Review text problem C-4.)

p-2 *ADDING VECTORS (CAP. 3):* (a) On the following grid, construct an arrow representing the sum $\vec{X} = \vec{Y} + \vec{Z}$ of the vectors \vec{Y} and \vec{Z} . ▶



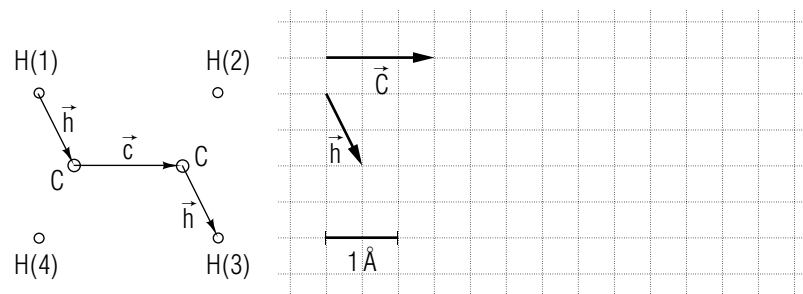
(b) Express \vec{X} using one of the unit vectors \hat{x} and \hat{y} . (Answer: 6) (Suggestion: Return to text problem D-1 and make sure your work is correct.)

p-3 *ADDING VECTORS (CAP. 3):* For each of the vector pairs \vec{A} and \vec{B} on this grid:



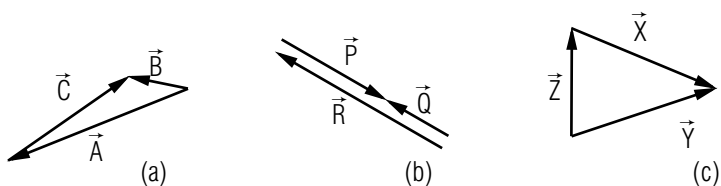
What is the magnitude $|\vec{S}|$ of each vector sum? What is the numerical sum $S = A + B$ of each pair of magnitudes A and B ? (Answer: 12) (Suggestion: Review text problem D-3.)

p-4 *ADDING VECTORS (CAP. 3):* The organic molecule ethylene consists of six atoms all lying in a plane as indicated in the following drawing. The displacement \vec{d} from hydrogen atom H(1) to H(3) is $\vec{d} = \vec{h} + \vec{c} + \vec{h} = \vec{c} + 2\vec{h}$. On the following grid, construct an arrow representing the displacement \vec{d} . ▶

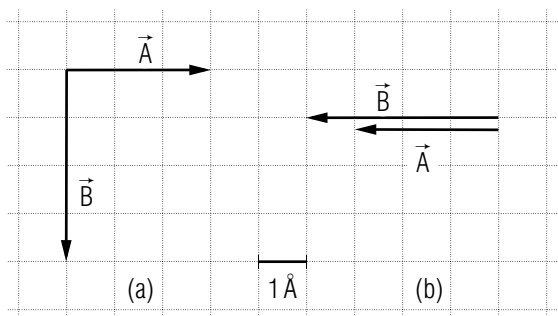


What is the distance d between Hydrogen atoms (1) and (3)? (Answer: 14) (Suggestion: Review text problem D-4.)

p-5 *GRAPHIC TO ALGEBRAIC SYMBOLS (CAP. 4):* For each of the following diagrams, write an algebraic equation expressing one of the three vectors as the sum of the other two. (Answer: 11) (Suggestion: Review text problem D-6.)

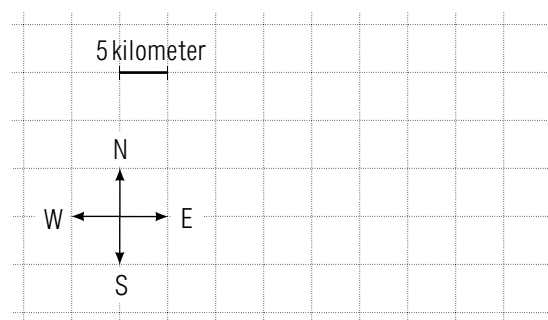


p-6 *SUBTRACTING VECTORS (CAP. 3):* Each of the pairs of vectors \vec{A} and \vec{B} shown in the following diagrams represent possible position vectors of two atoms in a molecule relative to an origin 0. For each pair of vectors, draw an arrow representing the vector difference $\vec{D} = \vec{B} - \vec{A}$. ▶



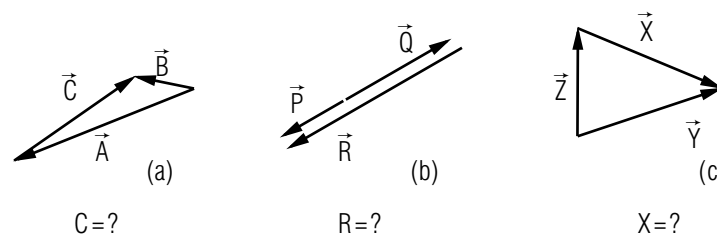
What is the magnitude D of each vector difference (the distance between the two atoms in the molecule)? What is the numerical difference $B - A$ between the two magnitudes B and A ? (*Answer: 13*) (*Suggestion: Review text problem E-3.*)

p-7 *SUBTRACTING VECTORS (CAP. 3):* Using the spots on his radar screen, a radar operator records that a ship has an initial position vector $\vec{r}_1 = 20$ kilometer west of the radar station, and that after a time interval $T = 0.5$ hour, the ship has a position vector $\vec{r}_2 = 15$ kilometer south of the radar station. On the following grid construct an arrow representing the displacement $\vec{D} = \vec{r}_2 - \vec{r}_1$ of the ship during this time interval. ▶



Use the approximate relation $\vec{V} = \vec{D}/T$ to find the magnitude V of the ship's velocity. (*Answer: 15*) (*Suggestion: Review text problem E-4.*)

p-8 *GRAPHIC TO ALGEBRAIC SYMBOLS (CAP. 5):* For each of the following vector diagrams, write an algebraic equation expressing the indicated vector as the difference of the other two. (*Answer: 9*) (*Suggestion: Review text problem E-5.*)



SUGGESTIONS

s-1 (Text problem F-2): To find the value of \vec{v}_B , use your algebraic expression for this quantity, and carefully substitute the known values including signs, units, and unit vectors. Most errors in manipulating vector equations come from careless substitution or careless arithmetic of signed numbers.

s-2 (Text problem B-4): A vector quantity must include a direction. A sign (+5 meter, -200 foot) does not specify a direction, nor does a description of where the quantity is measured (50 meter between two trees). To decide whether a quantity is a vector or not, it may help to ask Does this quantity include a direction in which I could point my finger?

s-3 (Text problem E-4): Arrow symbols almost always help in understanding the relation between vectors. Draw arrow symbols for \vec{B} and \vec{W} with their tails together. Use the definition of subtraction to construct an arrow representing the difference $\vec{D} = \vec{W} - \vec{B}$.

s-4 (Text problem D-6): According to its definition, the vector which is the sum of the other two must have its tail at the tail of one vector and its tip at the tip of the other.

s-5 (Text problem E-5): You may find it easiest to first use the definition of vector addition to write an equation relating the three vectors. For example, $\vec{A} = \vec{C} + \vec{B}$ in the first diagram. This equation can then easily be solved for the indicated vector \vec{B} .

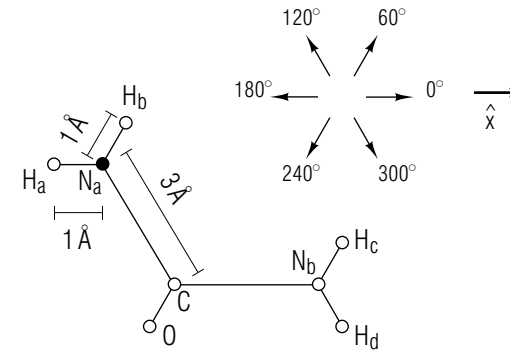
s-6 (Text problem E-1): Follow the directions of the definition of subtraction: draw arrow symbols for \vec{v}_A and \vec{v}_B with their tails together. (Remember that the arrow symbol for a vector can be drawn in any location, so long as it has the correct length and direction.) Then an arrow representing $\Delta\vec{v} = \vec{v}_B - \vec{v}_A$ can be drawn from the tip of \vec{v}_A to the tip of \vec{v}_B . To check your diagram, rearrange the previous equation to get a relation involving addition, such as: $\Delta\vec{v} + \vec{v}_A = \vec{v}_B$. Then use your knowledge of vector addition to check that your diagram correctly represents this relation.

Finally express the vector $\Delta\vec{v}$ as a multiple (positive or negative) of the parallel unit vector \hat{y} .

In finding the change $v_B - v_A$, remember that the speeds v_A and v_B are equal.

s-7 (Text problem C-4): When vectors become confusing, it almost always helps to draw arrow symbols. On the following diagram, draw and label arrows for the vectors of interest: \vec{X} , the displacement from N_a to H_a ; \vec{Y} , the displacement from N_a to H_b ; and \vec{Z} , the displacement from N_a to C .

►



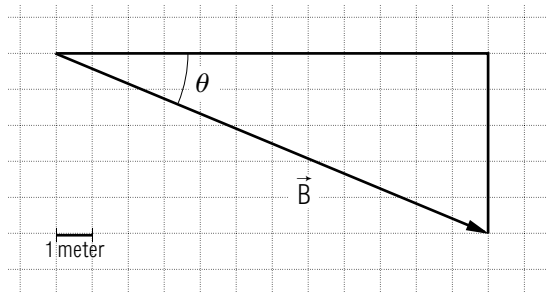
Values for \vec{X} , \vec{Y} , and \vec{Z} must include directions. To specify directions, compare the arrows you have drawn with the six arrows used to indicate directions. Two arrows have the same direction if they are parallel and point the same way.

(Answer: 8)

s-8 (Text problem E-3): The difference $L - M$ in the magnitudes L and M is just the difference of the two numbers 40 km and 30 km. This is not the same as the magnitude $|\vec{D}|$ of the vector difference $\vec{L} - \vec{M}$, which can have a variety of values depending on the directions of \vec{L} and \vec{M} .

Vector diagrams involving parallel vectors can be confusing if the arrows are drawn along a single line so that they lie on top of one another. It is helpful to use the procedure described in Suggestion [s-12] which suggests drawing such vectors parallel, but slightly separated.

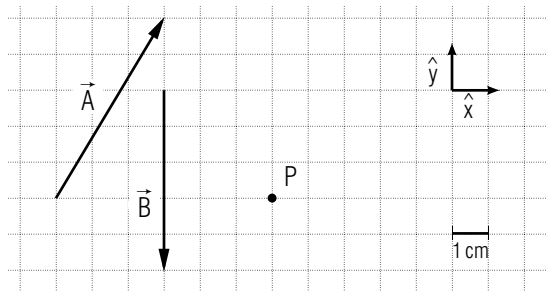
s-9 (Text problem B-2): Part (b): To find the magnitude B and the angle θ , consider the right triangle shown in this drawing:



The shorter sides of this triangle can be found by counting the grid spacings. Then since \vec{B} forms the hypotenuse of this triangle, use the Pythagorean theorem to find the length B . Then find the angle θ by using the definition $\tan \theta = (5 \text{ meter})/(12 \text{ meter})$.

s-10 (Tutorial frame [c-5]): We have defined only the division of a vector by a number. Remember that division of a vector is by a number n is the same as multiplication of the vector by $(1/n)$.

s-11 (Text problem D-1): In using arrow symbols to add vectors, it is crucial to use the fact that an arrow representing a vector can be drawn in *any* position. To provide yourself with working space, draw an arrow representing \vec{A} beginning at the point P on this grid:



According to the definition of addition of vectors, to add two vectors, \vec{A} and \vec{B} , the arrow representing \vec{B} should be drawn from the tip of the arrow representing \vec{A} .

On the preceding grid, draw an arrow representing \vec{B} in this way.

►

When arrows representing \vec{A} and \vec{B} are drawn in this way, an arrow representing $\vec{S} = \vec{A} + \vec{B}$ can be drawn from the tail of \vec{A} to the tip of \vec{B} . On the preceding grid, draw an arrow representing the sum \vec{S} .

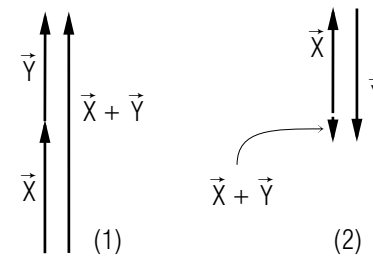
►

(Answer: 10) Now: Go to practice problem [p-2].

s-12 (Text problem D-3): The sum of the magnitudes, $X + Y$, is just the sum, of the numbers 0.3 km and 0.4 km. This is not the same as the magnitude $|\vec{S}|$ of the vector sum $\vec{S} = \vec{X} + \vec{Y}$, which has a variety of values depending on the directions of \vec{X} and \vec{Y} .

Now: If you need help in adding parallel vectors, continue. Otherwise, return to text problem D-3.

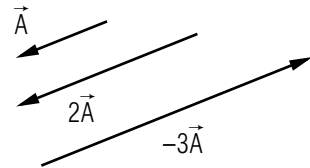
Arrow diagrams representing addition of parallel vectors can be confusing since several arrows may lie on top of one another. For example, according to the definition of vector addition, the arrow representing the sum $\vec{X} + \vec{Y}$ of the vectors in the following diagram (1) should be drawn from the tail of \vec{X} to the tip of \vec{Y} . However, to avoid covering up the arrows representing \vec{X} and \vec{Y} , you may prefer to draw the arrow representing the sum slightly to the right (or left) of the original arrows, as indicated here.



To add the vectors \vec{X} and \vec{Y} in the preceding diagram (2), the vector \vec{Y} should be drawn from the tip of the arrow representing \vec{X} . However, to avoid covering \vec{X} , the arrow representing \vec{Y} has been drawn slightly to the right of \vec{X} . Then the arrow representing the sum $\vec{X} + \vec{Y}$ is drawn from the tail of \vec{X} to a point slightly to the left of the tip of \vec{Y} .

ANSWERS TO PROBLEMS

1. The arrow $2\vec{A}$ must have the same direction as the arrow \vec{A} and be twice as long. The arrow $-3\vec{A}$ must have the opposite direction to \vec{A} and be 3 times as long. For example, you might have drawn this:

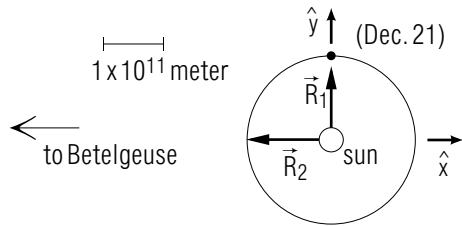


2.

	\vec{A}	\hat{x}
SI unit:	meter/sec ²	none or 1
Magnitude:	3 meter/sec ²	1

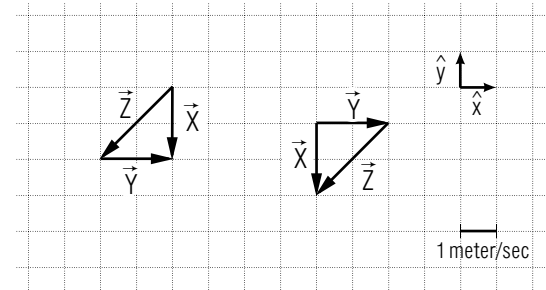
 ; \hat{y} and \hat{z} .

3. $\vec{D} = 2.0 \times 10^2$ meter west. Answer must include unit and direction.
 4. a.



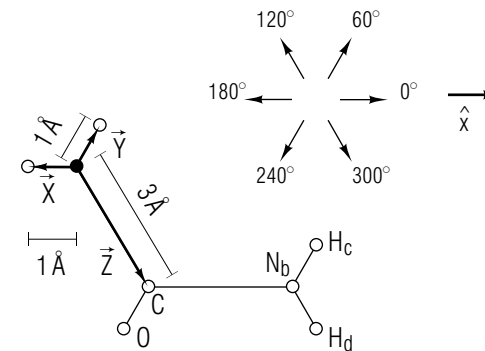
- b. $\vec{R}_1 = (1.5 \times 10^{11} \text{ meter})\hat{y}$
 c. $\vec{R}_2 = (-1.5 \times 10^{11} \text{ meter})\hat{x}$
 5. (1) Unit of $b\vec{X}$ is the unit of b times the unit of \vec{X} ; (2) (b)

6. a. Either of these constructions of \vec{X} :



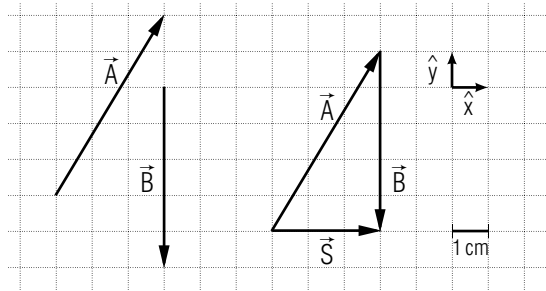
- b. $\vec{X} = (-2 \text{ meter/sec})\hat{y}$. (Check unit, sign, and direction)

7. $\vec{V}T = 10$ meter east; $\vec{D}/T = 3.0$ meter/sec north; $\vec{D}\vec{V}$ cannot be found.
 8.



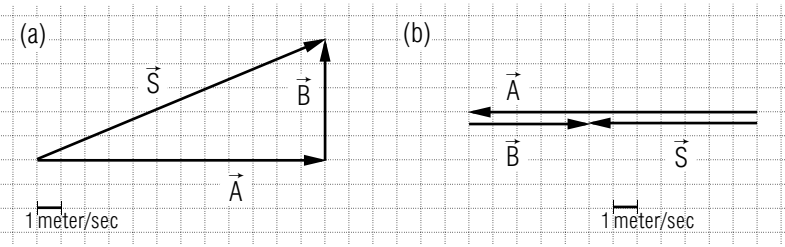
9. a. $\vec{C} = \vec{B} - \vec{A}$
 b. $\vec{R} = \vec{P} - \vec{Q}$
 c. $\vec{X} = \vec{Y} - \vec{Z}$

10.



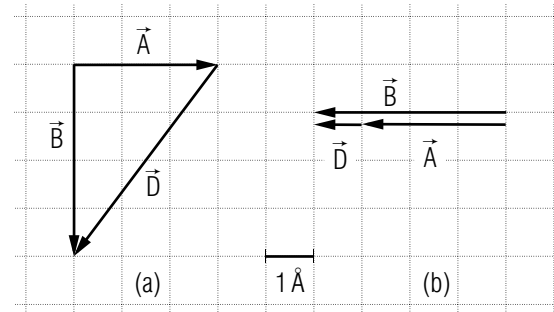
11. a. $\vec{B} = \vec{A} + \vec{C}$
 b. $\vec{Q} = \vec{R} + \vec{P}$
 c. $\vec{Y} = \vec{Z} + \vec{X}$ (In each case, the two added vectors may have the opposite order.)

12.



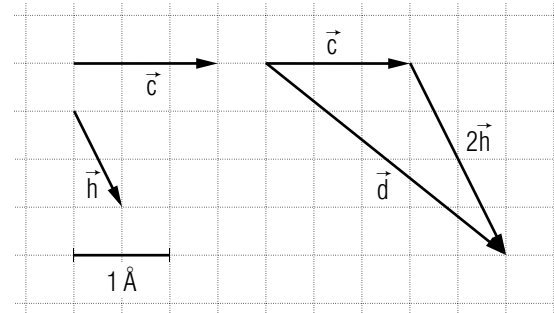
- a. $|\vec{S}| = 13$ meter/sec, $A + B = 17$ meter/sec
 b. $|\vec{S}| = 7$ meter/sec, $A + B = 17$ meter/sec

13.



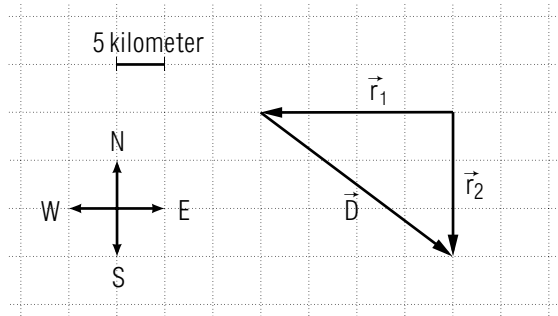
- a. $D = 5$ angstrom, $B - A = 1$ angstrom
 b. $D = 1$ angstrom, $B - A = 1$ angstrom

14.



$$d = \sqrt{2.5^2 + 2.0^2} \text{ angstrom} = 3.2 \text{ angstrom}$$

15.

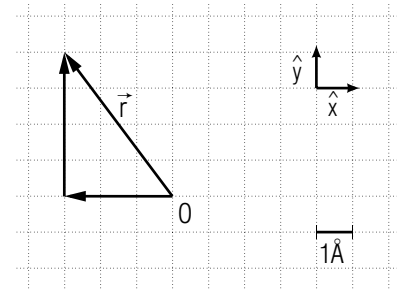


$$V = 50 \text{ km/hr}$$

101. a. 70 meter east, 70 meter along \hat{x}
 b. 100 meter north west, 100 meter opposite to \hat{k}
 c. 100 meter south east or 100 meter along \hat{k}
102. a. number negative 12 (degree Fahrenheit)
 b. vector upward 5,000 meter
 c. vector west 20 meter/sec
 d. number positive 2 meter
103. a. The arrow should be labeled by A, have a direction opposite to \hat{y} , and be six grid spacings in length.
 b. $B = 13 \text{ meter}$, $\theta = 23^\circ$
104. Magnitudes equal: (a), (c), (e), (f); Directions equal: (c), (d), (e); Vectors equal: (c), (e).
105. a. $\vec{a}_1 = 2 \text{ meter/sec}^2$ along \hat{x} , $\vec{a}_2 = \vec{a}_3 = 1 \text{ meter/sec}^2$ opposite to \hat{x} , $\vec{a}_4 = \vec{0}$ or 0 (Any value except $\vec{a}_4 = 0$ must include both direction and units.)
 b. $\vec{a}_2 = \vec{a}_3$
 c. 180°
106. No. (Magnitudes are not equal; they have different units.)
107. a. V , $|\vec{V}|$
 b. km/hour

- c. positive or zero (if $\vec{V} = 0$)
 d. magnitude $V = 4 \text{ km/hour}$

108. The arrow $3\vec{X}$ must have the *same* direction as the arrow \vec{X} , and be 3 times as long. The vector $-\vec{X}$ must have the *opposite* direction from \vec{X} and have the same length. The vector $(-2)\vec{X}$ must have the *opposite* direction from \vec{X} and be twice as long.
109. No.
110. a. $\vec{v} = (3.0 \text{ meter/sec})\hat{x}$
 b. $\vec{v} = (-0.30 \text{ meter/sec})\hat{x}$ (Units. and direction must be included.)
111. a. $\vec{S} = (0.5 \text{ km})\hat{z}$, $|\vec{S}| = 0.5 \text{ km}$, $\vec{X} + \vec{Y} = 0.7 \text{ km}$
 b. $\vec{S} = (0.7 \text{ km})\hat{y}$, $|\vec{S}| = 0.7 \text{ km}$, $\vec{X} + \vec{Y} = 0.7 \text{ km}$
 c. $\vec{S} = (-0.1 \text{ km})\hat{y}$, $|\vec{S}| = 0.1 \text{ km}$, $\vec{X} + \vec{Y} = 0.7 \text{ km}$
112. a. \vec{A} is 4 grid spacings in length, and directed towards the left (opposite to \hat{x}). \vec{B} is 2 grid spacings in length and directed towards the right (along \hat{x}).
 b. $\vec{C} = (3 \text{ km})\hat{y}$.
- 113.

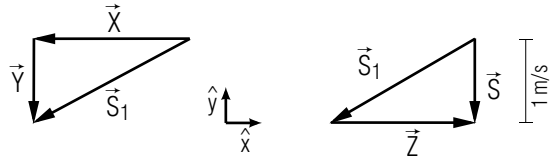


The horizontal vector is $(-3 \text{ \AA})\hat{x}$, and the vertical vector is $(4 \text{ \AA})\hat{y}$. Their sum is $\vec{r} = (-3 \text{ \AA})\hat{x} + (4 \text{ \AA})\hat{y}$.

- b. $r = 5 \text{ angstrom}$
 c. 120°
114. a. $\vec{X} = 1 \text{ \AA}$ in the direction 180° from \hat{x} , or $\vec{X} = (-1 \text{ \AA})\hat{x}$; $\vec{Y} = 1 \text{ \AA}$ in the direction 60° from \hat{x} ; $\vec{Z} = 3 \text{ \AA}$ in the direction 300° from \hat{x}

b. $\vec{X} = (-1 \text{ \AA})\hat{x}$

115. a.



$\vec{S} = (-1 \text{ meter/sec})\hat{y}$

b. $\vec{S}_2 = \vec{X} + \vec{Z} = 0$, $\vec{S} = 0 + \vec{Y} = (-1 \text{ meter/sec})\hat{y}$

116. a. The vector \vec{B} begins at the tip of \vec{A} , and \vec{S} is drawn from the tail of \vec{A} to the tip of \vec{B} .

b. The vector \vec{A} begins at the tip of \vec{B} , and \vec{S}' is drawn from the tail of \vec{B} to the tip of \vec{A} .

c. $\vec{S} = \vec{S}' = (3 \text{ cm})\hat{x}$

117. $\vec{B} = \vec{C} + \vec{A}$, $\vec{Y} = \vec{Z} + \vec{X}$

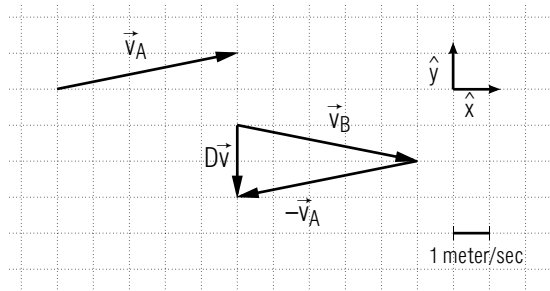
118. a. $\vec{v}_B = \vec{a}t + \vec{v}_A$

b. $\vec{v}_B = (-15 \text{ meter/sec})\hat{y}$

119. $\Delta\vec{v} = (-2 \text{ meter/sec})\hat{y}$, $v_B - v_A = 0$

120. $\vec{B} = \vec{A} - \vec{C}$, $\vec{Z} = \vec{X} - \vec{Y}$

121.



Yes

122. $\vec{C} = (-1 \text{ meter/sec}^2)\hat{x}$. No, $C = 1 \text{ meter/sec}^2$ is not equal to $A+2B = 11 \text{ meter/sec}^2$.

123. a. $\vec{D} = (+50 \text{ km})\hat{D}$, $|\vec{D}| = 50 \text{ km}$, $L - M = 10 \text{ km}$, where \hat{D} is a unit vector pointing directly away from the tip of \vec{M} and directly toward the tip of \vec{L} on the graph.

b. $\vec{D} = (-10 \text{ km})\hat{y}$, $|\vec{D}| = 10 \text{ km}$, $L - M = 10 \text{ km}$

c. $\vec{D} = (-70 \text{ km})\hat{x}$, $|\vec{D}| = 70 \text{ km}$, $L - M = 10 \text{ km}$

124. a. $S_1 = 5 \text{ meter}$, $S_1^2 = A^2 + B^2$

b. $S = 13 \text{ meter}$, $S^2 = S_1^2 + C^2$

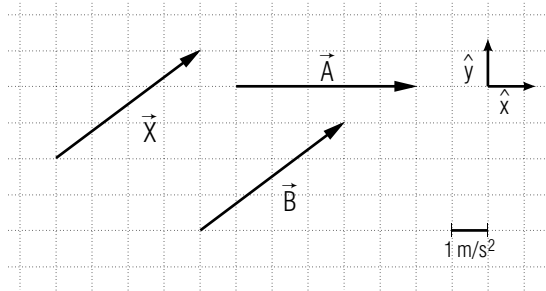
c. $S^2 = A^2 + B^2 + C^2$

125. $\vec{D} = 630 \text{ km southwest}$, $\vec{V} = 70 \text{ km/hr southwest}$. (Check directions and units.)

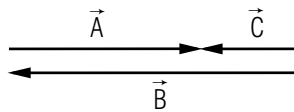
126. $d = \sqrt{12} \text{ angstrom} \approx 3.5 \text{ angstrom}$

MODEL EXAM

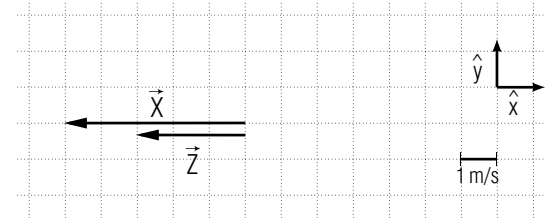
1. *Comparing vectors.* Use this diagram:



- a. What is the magnitude of the vector \vec{X} ?
 - b. Compare the vectors \vec{X} and \vec{A} . Are their magnitudes equal? Are their directions the same? Are these vectors equal?
 - c. Compare the vectors \vec{X} and \vec{B} . Are their magnitudes equal? Are their directions the same? Are these vectors equal?
2. *Positions of a ship.* The radar operator at a lighthouse records that a ship has a position vector $\vec{r}_1 = 500$ meter north of the lighthouse. At a later time he records that the position vector of the ship is $\vec{r}_2 = (-1200$ meter) \hat{x} , (where \hat{x} points east). Draw arrows representing the vectors \vec{r}_1 and \vec{r}_2 . Then construct an arrow representing the displacement $\vec{D} = \vec{r}_2 - \vec{r}_1$.
3. *Translating vector relations.* Using the vectors shown in the following diagram to write an expression for \vec{C} in terms of \vec{A} and \vec{B} .



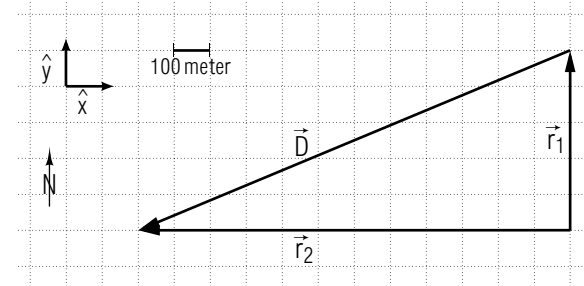
4. *Solving vector equations.* The equation $\vec{X} = 2\vec{Y} - \vec{Z}$ relates the vector \vec{Y} to the vectors \vec{X} and \vec{Z} shown in this diagram:



- a. Express \vec{Y} in terms of the known vectors \vec{X} and \vec{Z} .
- b. What is the value of the vector \vec{Y} ?

Brief Answers:

1. a. 5 m/s^2
 - b. magnitudes: equal or yes; directions: different or no; vectors: not equal or no
 - c. magnitudes: equal or yes; directions: same or yes; vectors: equal or yes
- 2.



Arrows representing \vec{r}_1 and \vec{r}_2 ; Arrow representing \vec{D}

3. $\vec{C} = \vec{A} + \vec{B}$
4. a. $\vec{Y} = (\vec{X} + \vec{Z})/2$
- b. $\vec{Y} = (-4 \text{ m/s}) \hat{x}$