PHYSNET

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FORMAL STRUCTURE OF QM (II)

Quantum Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

FORMAL STRUCTURE OF QM (II)

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Title: **Formal Structure of QM (II)**

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Input Skills:

1. Review the properties of vector spaces from your studies of linear algebra.
2. Unknown: assume (MISN-0-388).

Output Skills (Knowledge):

K1. Vocabulary: linearly independent, basis, orthonormal basis, inner or scalar product, norm of a vector.

K2. Prove that the representation of an arbitrary vector on a given basis is unique.

K3. Show that the inner product of a vector with itself is positive definite.

K4. Prove the Schwartz inequality.

Output Skills (Problem Solving):

S1. Perform exercises involving the expansion of wave functions on an orthonormal basis such as those given in the procedure.

External Resources (Required):

1. E. E. Anderson, *Modern Physics and Quantum Mechanics*, Saunders (1971).

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1. Introduction

In this unit we shall examine the relationship between arbitrary wave functions and eigenstates of hermitian operators. This is best achieved by introducing the concept of an infinite-dimensional linear scalar product space or "Hilbert Space."

2. Procedures

1. Read section 6.3 through the middle of page 216. In addition, you may wish to review the properties of vector spaces from your studies of linear algebra. See also the top of page 217 for the last definition.

   The distinguishing characteristics of the Hilbert Spaces of quantum mechanics are:

   a. The *vectors* represent the *states* of the system. In the coordinate representation, the states are represented by wave functions.

   b. An arbitrary vector can be expressed as a *linear combination* of *basis* vectors. These basis vectors are usually chosen to be the eigenstates of a hermitian operator, and most of the time of the Hamiltonian.

   There is a theorem which we shall not prove that states:

   > For any hermitian operator Q, there exists at least one basis in the Hilbert Space such that each function is an eigenfunction of Q.

   This is a very important theorem which says that the eigenfunctions of Q are *complete*, i.e. a basis can be formed from them so that every state can be expanded in terms of eigenfunctions of Q. We shall return to the meaning of these expansions later.

   A basis usually has infinitely many members - e.g. the infinite square well eigenfunctions form a basis for the states of "a particle in a 1 - dimensional box." All members of the basis must be *linearly independent*, see top of page 216. If not, 1 member of the basis could be

   expressed in terms of the others and so would be unnecessary. Intuitively, a basis is the largest set of linearly independent vectors one can extract from the space, as well as the smallest set of vectors that one can use to construct an expansion for any vector. The number of vectors in the basis is called the *dimension* of the space - usually infinite.

   (c) It is usual to use an orthonormal basis, for which $< \hat{\psi}_i|\hat{\psi}_j >= \delta_{ij}$, where $\hat{\psi}_i$ and $\hat{\psi}_j$ are any two members of the basis. Recall that $\hat{\psi}_i$ and $\hat{\psi}_j$ will automatically be orthogonal if they correspond to different eigenvalues. If $\hat{\psi}_i$ and $\hat{\psi}_j$ are degenerate, the Schmidt orthogonalization procedure can be used; but we will not take that up.

2. Read the proof in the middle of page 216 and be sure you can reproduce it. Note that this does *not* depend on whether or not the $\hat{\psi}_i$ form an *orthonormal* basis.

3. Read the proof at the bottom of page 216 and be sure you can reproduce it. This proof assumes the $\hat{\psi}_i$ are orthonormal. The positive definiteness of $< \psi|\psi >$ is usually considered to be part of the *definition* of a scalar product.

4. Read the proof on page 217 and be sure you can reproduce it. This very useful inequality is often used in theoretical discussions. Under what conditions does equality hold?

5. Consider a particle in an infinite square well in one-dimension with walls at $x = \pm\alpha$.

   a. Write down the correctly normalized wave functions. These form an infinite orthonormal basis for this system.

   b. The wave function to be studied is

   $$\psi(x) = A\cos\left(\frac{\pi}{\alpha}x\right)\cos\left(\frac{\pi}{2\alpha}x\right)$$

   Normalize this wave-function, i.e. determine A.

   c. Express $\psi(x)$ as a linear combination of basis vectors. To do this, prove and use the following theorem:
   Let $\hat{\psi}_i$ be an orthonormal basis and let $\psi = \sum_i C_i\hat{\psi}_i$. Then:

   $$C_i =< \hat{\psi}_i|\psi > \ .$$

d. Verify explicitly that $\sum_i |C_i|^2 = 1$ for your coefficients of part (c).

e. What is the probability of obtaining each energy eigenvalue as a result of an energy measurement? Prove that your probability is a number between 0 and 1.

f. Repeat steps (b) through (e) for $\psi(x) = A\cos^2(\pi x/2\alpha)$. You will now get an infinite number of C's. If you can't sum the series in (d) add the first 10 terms and see how close you come to 1.

Acknowledgments

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