PHYSNET

MISN-0-387

HARMONIC OSCILLATOR II

Quantum Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

HARMONIC OSCILLATOR II

by

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Title: **Harmonic Oscillator II**

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Input Skills:

1. Unknown: assume (MISN-0-386).

Output Skills (Knowledge):

K1. Write the time-independent Schrodinger equation for the harmonic oscillator in momentum space and the solutions corresponding to $n = 0$, $n = 1$, and $n = 2$.

K2. Write the $n = 0$, $n = 1$, and $n = 2$ eigenstates in coordinate space and explicitly verify that the coordinate space eigenfunctions are the Fourier transforms of the momentum space eigenfunctions and vice-versa.

K3. Define the raising and lowering operators $a\dagger$ and a.

Output Skills (Rule Application):

R1. Calculate the commutator of $a\dagger$ and a, write the Hamiltonian in terms of $a\dagger$ and a, and deduce from this the eigenvalues of $a\dagger$, a.

Output Skills (Problem Solving):

S1. Show that $H(a\psi_n) = (n - 1/2)\,\hbar\omega(a\psi_n)$ and $H(a\dagger\psi_n) = (n + 3/2)\,\hbar\omega(a\dagger\psi_n)$.

External Resources (Required):

1. E. E. Anderson, *Modern Physics and Quantum Mechanics*, W. B. Saunders (1971).

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New authors, reviewers and field testers are welcome.

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1. Introduction

At this stage we shall use the harmonic oscillator to illustrate many of the concepts we have developed in previous units. We will also introduce raising and lowering operators which are of great use in many areas of quantum mechanics.

2. Procedures

1. The Schrodinger equation is $(T+V)\psi = E\psi$. To express the equation in the momentum space representation, it is only necessary to express T and V in that representation:

$$T = \frac{p_x^2}{2m},\ V = \frac{1}{2}kx^2 = -\frac{1}{2}k\hbar^2\frac{\partial^2}{\partial p_x^2}$$

The equation is thus,

$$\frac{-\hbar^2 k}{2}\frac{\partial^2\phi}{\partial p_x^2} + \frac{p_x^2}{2m}\phi = E\phi$$

where ϕ is the time-independent wave function in momentum space.

2. a. The equation in procedure 1 is exactly of the same form as

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}kx^2\psi = E\psi$$

with k and $1/m$ interchanged. Use this observation and the known coordinate space wave-functions to write the *normalized* momentum space wave-functions for $n = 0, 1$ and 2. Substitute your momentum-space wave functions into the Schrodinger equation and verify that they are indeed the required eigenfunctions. Also verify the normalization explicitly for $n = 1$.

b. From your previous work, you should know how $\phi_n(p)$ and $\psi_n(x)$ are related by Fourier transformation. Write down the connection.

Verify this connection explicitly (in either direction) for $n = 0, 1$ and 2.

3. Define new dimensionless operators by

$$p \equiv p_x/\hbar\sqrt{\alpha} \text{ and } q \equiv \sqrt{\alpha}x \text{ where } \alpha \equiv m\omega/\hbar$$

The raising operator a+ is defined by

$$a^+ \equiv \frac{1}{\sqrt{2}}(q - ip);\ a \equiv \frac{1}{\sqrt{2}}(q + ip)$$

is the lowering operator.

4. Show that $[q, p] = i$. Show that $[a, a^+] = 1$. Express p and q in terms of a^+ and a. Show that the Hamiltonian is

$$1/2(p^2 + q^2)\hbar\omega = (a^+a + 1/2)\hbar\omega$$

Knowing the eigenvalues of $\mathcal{H}$, deduce the eigenvalues of a^+a. For this reason a^+a is sometimes called the "number operator." BEWARE of non-commuting operators in this computation.

5.

$$\begin{gathered}\mathcal{H}(a\psi_n) = (a^+a + 1/2)\hbar\omega(a\psi_n) = \\ \frac{1}{2}\hbar\omega a\psi_n + \hbar\omega a^+a^2\psi_n = \\ \frac{1}{2}\hbar\omega a\psi_n + \hbar\omega a(a^+a\psi_n) - \hbar\omega[a, a^+a]\psi_n = \\ \frac{1}{2}\hbar\omega a\psi_n + n\hbar\omega a\psi_n - \hbar\omega[a, a^+a]\psi_n\end{gathered}$$

Use the commutator of Output Skill R1 to evaluate the commutator and obtain the desired result. Follow a similar procedure for $\mathcal{H}(a^+\psi_n)$.

Acknowledgments

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