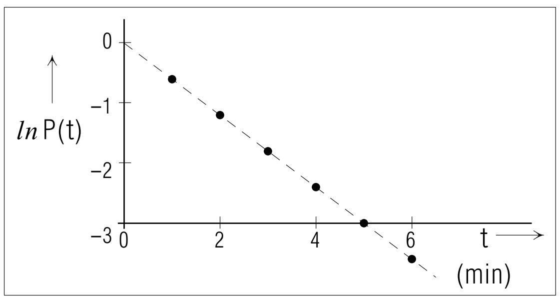
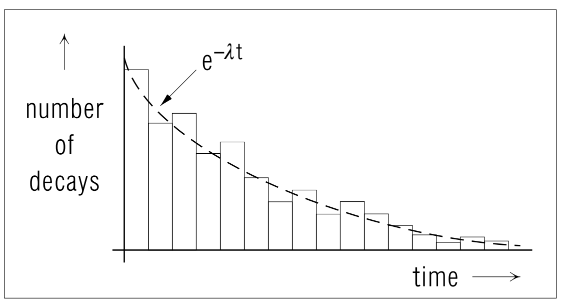
PHYSNET

MISN-0-311

EXPONENTIAL DECAY: OBSERVATION, DERIVATION





Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

EXPONENTIAL DECAY: OBSERVATION, DERIVATION

by

Peter Signell

Title: **Exponential Decay: Observation, Derivation**

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Input Skills:

1. Vocabulary: natural logarithm, exponential function, probability (MISN-0-401).
2. Perform simple definite integrals which lead to log functions, such as $\int_{x_0}^{x} dx'/x' = \ell n\,(x/x_0)$ (MISN-0-1).
3. Differentiate simple exponential functions (MISN-0-1). For example: if $f = \exp[-a(x - x_0)]$, show that $df/dx = -af$.
4. Given a graphed straight line, determine slope, intercept, and equation (MISN-0-401).

Output Skills (Knowledge):

K1. Describe the relationship between the exponential decay law and typical finite-number data.

K2. Derive the exponential decay law, for the number of undecayed systems, starting from the "no aging" (constancy of decay constant) assumption.

K3. Derive the exponential decay law, for the probability that a single system is undecayed, starting from the "no aging" (constancy of decay constant) assumption.

Output Skills (Rule Application):

R1. Determine whether or not a given set of decay data is consistent with an exponential description.

R2. Determine the mean life for a given set of exponential decay data.

External Resources (Required):

1. A source of natural logarithms (table or calculator) and several sheets of graph paper.

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1. Nuclear Decay: Exponential

1a. The Exponential Decay Law (EDL). Suppose we examine identically prepared systems belonging to a single nuclear species that decays radioactively. One of the first experiments which comes to mind is that of measuring the times at which the decays occur in the sample at hand. When the decays have finally ceased, one can count the decays that were recorded and thereby learn the original number of radioactive systems which were undecayed at the beginning of observation. Then by subtraction one can find the number of such undecayed systems left at any particular time during the observations, and this number can be plotted as a function of time (see Fig. 1). Tens of thousands of such experiments have been performed on a great variety of decaying species with varying numbers of initial systems, and these experiments have led to the Exponential Decay Law, here after abbreviated EDL.[1] With few exceptions, the data on radioactive decays appear to be in agreement with this statement of the EDL: The number of systems still undecayed at time t, designated $N(t)$, is a decaying exponential function of time:[2]

$$N(t) = N(0)e^{-\lambda t}\,. \qquad [Q-1] \qquad (1)$$

[1]Not a widely used abbreviation.

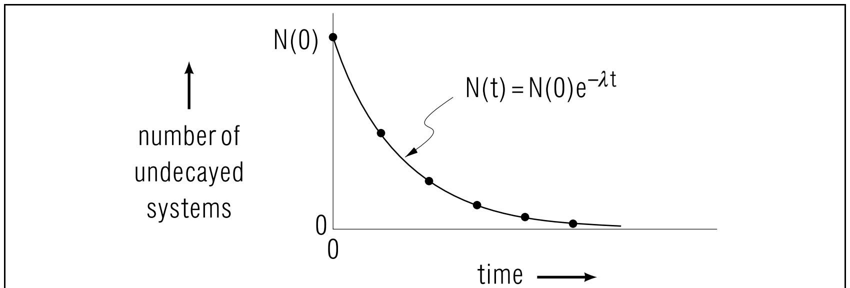

Figure 1. Typical experimental results for the decay of a large number of radioactive atoms having a unique decay mode.

The "decay constant," λ, is uninfluenced by environmental characteristics such as temperature and pressure: it is a characteristic number for each nuclear species.[3]

1b. Decay-Constant and Mean-Life Values. Values of the decay constant range from 1 decay in 10^{12} years for the alpha decay of Sm^{152} to 1 per 10^{-23} sec. for the strong decay of the rho meson. The number of undecayed systems at time zero is usually under the control of the experimenter. An exponentially-decaying species's mean lifetime, called its "mean life" $\bar{t}$, can be shown to be the inverse of its decay constant: $\bar{t} = 1/\lambda$. *Help: [S-5]*

1c. "Explanation" of the EDL: The 3-Line Derivation. Whenever we find a large number of systems obeying a simple mathematical relation, such as the EDL, we are immediately challenged to find an explanation, a way of looking at the systems such that it becomes obvious which systems in nature should obey the law and which not. How does one go about finding such an explanation in the present case?

One approach to finding explanations, one which is used in mechanics, is to study the relationship between quantities and their rates of change. This involves integration and differentiation of quantities that are functions of time. In the present case we first define a useful rate function, the decay rate R:

$$R(t) \equiv -\frac{dN(t)}{dt}\,. \qquad \textit{Help: [S-4]} \qquad (2)$$

Combining Eqs. (1) and (2) we can relate the rate of decay at time t to the number remaining at time t:

$$R(t) = \lambda N(t). \qquad \textit{Help: [S-1]} \qquad (3)$$

Voila! Here is the explanation!

> "At any particular time, the rate of decay is proportional solely to the number of systems available to decay at that time."

[2][Q-1] refers to Question 1 in the Special Assistance Supplement for this module.
[3]Except for one rare type of decay known as "K-capture."

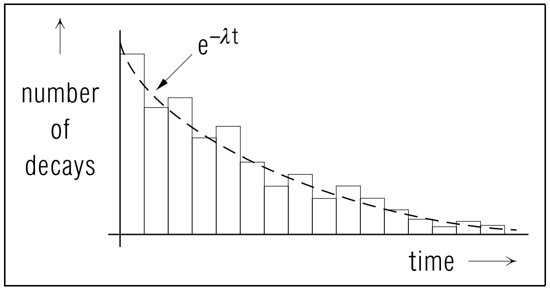

Figure 2. Typical results of decay measurements for a set of radioactive systems. The fluctuations are not due to experimental error.

This statement is usually assumed to be so obviously correct that it is taken as a basic explanation of the EDL. The EDL is generally derived from it in 3 lines. *Help: [S-2]* [Q1-3]

2. Comparison to Experiment

2a. Results of One Experiment. The exponential number and rate equations do not really coincide with what one usually finds experimentally. A typical set of experimental observations is shown in Fig. 2, where each vertical bar represents the total number of decays counted in the corresponding time interval or bin. Note that, far from obeying the monotonically decreasing exponential decay function of the EDL, some of the later bins have more counts than earlier bins. Furthermore if one took a complete second set of observations with the same original number of identically prepared systems, the data would not reproduce those of the first set. The tops of the bars for the second set would still fluctuate about the same smooth exponential curve but the fluctuations would be unpredictably different.

2b. Repetitions of the Same Experiment. If one repeated the Fig. 2 experiment many times and kept adding the new observations into the corresponding time bins, the vertical scale in Fig. 2 would have to be continually changed in order to keep the exponential curve in the same place on the graph paper. As the number of counts in each bin increased, the fluctuating appearance of the tops of the bins would decrease and the centers of the tops would appear to approach the smooth curve. Of course the fluctuations would still be there but they would look very small on the scale of the graph.

2c. N and R Equations Demand N, $R \to \infty$. It is now obvious that one must always write next to the exponential rate and number

equations; "valid only as $N(t) \to \infty$." However, it is possible to rewrite the equations and the basic assumption so as to be valid for any number of systems, even for a single decaying system. In order to do so, we will have to change to a probabilistic description. [Q4]

3. The Probabilistic EDL

3a. Probability Applied to Decay. The problem is to cast the EDL, and its rate equation, into a form that is valid for a finite number of systems and even for a single decaying system. The solution is to rewrite the EDL in terms of the probability $P(t)$ that any single specific system will still be undecayed at time t. Since all the decaying systems are identically prepared, each will have an identical time-evolving "undecay probability."

3b. The Probabilistic EDL. The probability of a single system being still undecayed at time (t) is defined experimentally and theoretically as the limit of the fraction of undecayed systems at that time:

$$P(t) = \lim_{N(0)\to\infty} \frac{N(t)}{N(0)}\,, \tag{4}$$

where $N(0)$ represents the total number of systems, decayed and undecayed. Applying this to the number and number rate equations, we get the "probabilistic exponential decay law" (EDL),

$$P(t) = e^{-\lambda t}\,, \tag{5}$$

and its rate equation,

$$r(t) \equiv -dP(t)/dt = \lambda e^{-\lambda t}\,. \tag{6}$$

Evaluating the law at time zero we find:

$$P(0) = 1.00\,, \tag{7}$$

which says that there is a 100% probability that the system is undecayed at time zero. "Time zero" is, of course, the time of creation of the system.

Thus our new exact statement of the basic assumption is:

> "The rate of decay of the probability P(t) is, at any time, proportional to the undecay probability P(t) remaining at that time."

As an equation, this basic assumption is written:

$$r(t) = \lambda P(t) \qquad (8)$$

which can be integrated in 3-6 lines to yield the probabilistic EDL. *Help: [S-3]* [Q5-9]

3c. Constant λ: No Internal Clock. The basic EDL assumption, stated above, carries the implication that the proportionality constant λ is independent of time. This means that all times are the same to an undecayed system; that its character remains completely unchanged until the instant of its decay. Can you swallow that? Usually, if something is going to decay it must have an internal clock of some kind which runs down: it must "age" until that cataclysmic moment when the vital element stops. Without such an aging process, how can an EDL system decay at all? The 3-line EDL derivation gives no answer; for that one must look to the field of physics called Quantum Mechanics.

3d. The "No-Aging" Assumption and QM. The mechanism of radioactive decay is generally described by Quantum Mechanics, which views "strictly exponential" decay as never more than a mathematical approximation to a certain segment in the life of a radioactive system. For example, the standard model of α-decay uses Quantum Mechanics' "Time-Dependent Schrödinger Equation." This equation nicely produces a time-evolving probability for all except completely stable (non-decaying) systems. Thus for decaying states it always violates the "no-aging" assumption of the 3-line EDL derivation. Nevertheless, the Time-Dependent Schrödinger Equation does produce the almost-exponential behavior observed for appropriate segments in the lives of appropriate systems.[4] Most radioactive systems of use in industry and basic research have enormous periods of time over which the EDL is an extremely accurate approximation and the "aging" is extremely small.[5]

4. Deducing Mean Life From Data

4a. The Semi-Log Plot. The first step in analyzing decay data is to make a semi-log plot. This means that a graph is made in which the

[4]See "Quantum Tunneling Through a Barrier" (MISN-0-250) for a pictorial presentation of the three main segments in the lives of quasi-stable systems. The pictures and film were generated by solving the Time-Dependent Schrödinger Equation on a computer.

[5]To obtain a feeling for the conditions under which this happens, see "A Soluble Model of Radioactive Decay" (MISN-0-312, Under Construction).

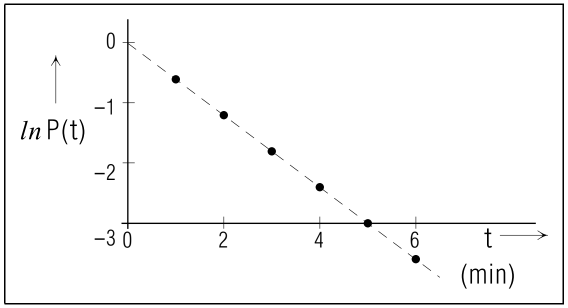

Figure 3. Illustrative example of logarithmic decay as seen on a semi-log plot. The mean life is $\bar{t}$ = 1.67 min.

vertical axis is effectively the logarithm of the dependent variable while the horizontal axis is the independent variable. This is accomplished by plotting each datum on a piece of semi-log graph paper or, equivalently, by plotting the logarithm of each datum on linear graph paper as in Fig. 3. If the data fall on a straight line, the decay is exponential.[6]

4b. Mean Life from Number Data. For exponential decay data where one plots the number of undecayed systems versus time, the mean life is just the negative inverse of the semi-log plot's slope:

$$\bar{t} = 1/\lambda\,; \qquad \lambda \equiv -d(\ell n\ N)/dt\,.$$

You can easily use these formulas to check the value given in the caption for Fig. 3. *Help: [S-9]*

4c. Mean Life from Rate Data. The mean life can be deduced from decay-rate data in a manner exactly parallel to that for number-of-undecayed-systems data. This is because the basic equation for the decay rate, $R(t) = -dN/dt$, has exactly the same mathematical form as Eq. 1 for the number $N(t)$. *Help: [S-10]*. Then the decay constant λ can be obtained immediately from a plot of $[\ell n\ R(t)]$ vs $[t]$, and $\bar{t}$ follows from λ.[7]

4d. Dealing with Data Scatter. Real-life data do not fall on a complete smooth curve but rather show a random-looking scatter about such a curve (see Fig. 2), and this must be dealt with in deducing a mean life. The scatter itself has two unrelated origins: experimental uncertainties or "errors," and the ever-present quantum fluctuations. The effect of the

[6]For a quick examination of the relationship between exponentiality and semi-log plots, see Appendix A, this module.

[7]Practical applications almost always use rate data: see "Some Uses of Radioactivity" (MISN-0-252), in which standard units of radioactivity measurement are introduced and used.

former can usually be reduced through better and/or more costly experimental design, but the effect of quantum fluctuations can be reduced only by increasing the number of data. Methods for dealing with such data-scatter are presented elsewhere,[8] but a simple result is that the deduced mean life value is made uncertain. Graphically speaking, the random character of the fluctuations make uncertain the location of the line on the semi-log plot. That uncertainty is transferred to the line's slope and hence to the mean life. In many experimental situations, however, the data are so large in number that they make λ and $\bar{t}$ accurate enough for practical applications.

Acknowledgments

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A. Semi-Log Plots and Exponential Behavior

An exponential decay function produces a straight line on a semi-log plot, and the slope of that line is the negative of the decay constant λ where, for example,

$$f(x) = f(0)\, \exp(-\lambda x)\,.$$

To prove these relationships, we take the natural logarithm of the above equation, getting:

$$\ell n\, f(x) = \ell n\, f(0) - \lambda x\,.$$ *Help: [S-11]*

We now define a new dependent variable,

$$y(x) \equiv \ell n\ f(x)\,,$$

which linearizes the relationship between the new dependent and independent variables:

$$y(x) = y(0) - \lambda x\,.$$

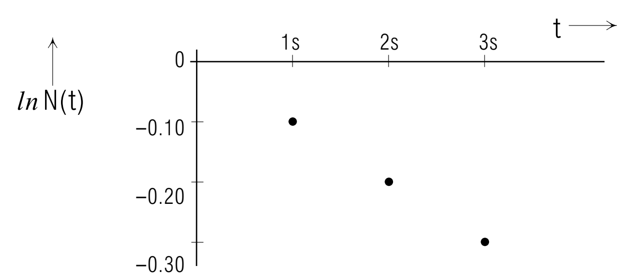
Thus plotting y vs. x on a graph produces a straight line whose slope is $-\lambda$.

[8] See "Linear Least Squares Fits to Data" (MISN-0-162, UC).

SPECIAL ASSISTANCE SUPPLEMENT

QUESTIONS:

1. By direct substitution of $t = 0$ in Eq. 1, determine and check the meaning of $N(0)$.
2. Why does Eq. 2 have a minus sign on the right hand side?
3. Verify that Eq. 3 follows from Eqs. 1 and 2.
4. Explain the correspondence between data and the exponential decay law, including time-bin fluctuations and the changes in their significance as the number of systems is increased.
5. Justify Eq. 4 qualitatively.
6. Show that Eq. 5 follows from the preceding equations.
7. Verify Eq. 6.
8. Verify Eq. 7.
9. Verify Eq. 8.
10. Sketch a curve, on a linear plot, which is an exponential function, and a curve which is not. On a semi-log plot, sketch a curve which represents an exponential function, and one which does not.
11. From the observations shown, determine the mean life for the species being observed. *Help: [S-6]*



12. From the three rate observations reported below, show graphically that the data are in agreement with exponential decay. Determine the species's mean life. [S-8]

t	$R(t)$
0 s	2638/sec
1 s	755/sec
2 s	216/sec

Brief Answers:

11. 10 s

12. 0.799 s

SEQUENCES:

S-1 *(from TX, 1c and [S-2])*

$$R(t) \equiv -\frac{dN(t)}{dt} = -\frac{d}{dt}\left[N(0)e^{-\lambda t}\right] = -N(0)\frac{d}{dt}e^{-\lambda t}$$
$$= -N(0)(-\lambda)e^{-\lambda t} = \lambda N(0)e^{-\lambda t} = \lambda\, N(t)$$

S-2 *(from TX,1c)*

Input:

(i) Assumption: Decay rate is proportional to number remaining. Stated as an equation: $R(t) = \lambda N(t)$. $\lambda \equiv$ const. of proportionality.

(ii) Rate-number relationship: $R(t) \equiv -\dfrac{dN(t)}{dt}$

3-Line Derivation:
Combining the input, $dN(t)/dt = -\lambda N(t)$, solution is: $N(t) = c\exp(-\lambda t)$ where c is the single constant of integration. Proof: by differentiation. *Help: [S-1]*
Substitute $t = 0$ and find $c = N(0)$ hence: $N(t) = N(0)\exp(-\lambda t)$

Alternative 6-Line Derivation:

$$\frac{dN(t)}{dt} = -\lambda N(t)$$
$$\frac{dN}{N} = -\lambda\, dt$$
$$\ell n\, N(t) - \ell n\, N(0) = -\lambda(t - 0)$$
$$\ell n \left(\frac{N(t)}{N(0)}\right) = -\lambda t$$
$$\frac{N(t)}{N90)} = e^{-\lambda t}$$
$$\Rightarrow N(t) = N(0)e^{-\lambda t}$$

S-3 *(from TX, 3b)*

Use $P(t) = \dfrac{N(t)}{N(0)}$ and $r \equiv -\dfrac{dP}{dt}$ in the Input and Derivation steps of [S-2], converting all N's and R's to P's and r's. Finally, use the 100% probabilty of being undecayed at time zero.

S-4 *(from TX, 4c)*

For decay, dN/dt is negative whereas the decay rate, the number of decays per unit time, is positive.
Mathematically: $N(t)$ is the number of undecayed particles at time t. Then the number of decayed particles is: $N_D(t) = N(0) - N(t)$. The rate of decay is:
$R(t) = \frac{dN_D(t)}{dt} = -\frac{dN(t)}{dt}$

S-5 *(from TX, 1b)*

The mean value of a species's lifetime is produced by weighting each time t with the number of systems $N(t)$ enjoying that time as part of their undecayed lives:

$$\bar{t} = \frac{\int_0^\infty tN(t)\,dt}{\int_0^\infty N(t)\,dt}$$

Thus $\bar{t} = 1/\lambda$ when $N(t) = N(0)\,\exp(-\lambda t)$. *Help: [S-7]*

S-6 *(from AS, [Q-11])*

$$\text{Slope} = \frac{\Delta(\ell n\, N)}{\Delta t} = \frac{-.10}{1s} = -0.10/\text{sec} = -\lambda. \qquad \Rightarrow \bar{t} = \frac{1}{\lambda} = 10\,\text{s}.$$

S-7 *(from AS, [S-5])*

Substitute $N(t)$ into the integrals and integrate. Note that:

$$\int_0^\infty te^{-\lambda t}\,dt = -\frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}\,dt = -\frac{d}{d\lambda}\left(\frac{1}{\lambda}\right) = \lambda^{-2}$$

You should remember this trick as well as the value of $\int_0^\infty \exp(-\lambda t)\,dt$.

S-8 *(from AS, [Q-12])*

$\ell n\,2638 = 7.878$
> difference $= -1.251$

$\ell n\,755 = 6.627$
> difference $= -1.251$

$\ell n\,216 = 5.375$

$$\text{slope} = \frac{\Delta(\ell n\,R)}{\Delta t} = \frac{-1.251}{1\,\text{s}} = -\lambda \qquad \Rightarrow \bar{t} = 1/\lambda = 0.799\,\text{s}.$$

S-9 *(from TX, 4b)*

$$\text{slope} = \frac{\Delta(\ell n\,P)}{\Delta t} = \frac{-3}{5\,\text{min}} = -\lambda \qquad \Rightarrow \bar{t} = 1/\lambda = 1.67\,\text{min}$$

S-10 *(from TX, 4c)*

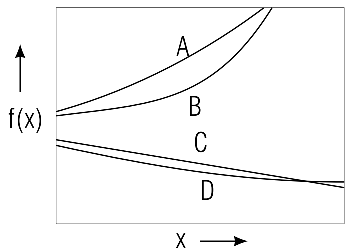
$N(t) = N(0)\,\exp(-\lambda t)$
$R(t) = -dN(t)/dt = \lambda N(0)\exp(-\lambda t)$
Then: $R(0) = \lambda N(0)$
Hence: $R(t) = R(0)\exp(-\lambda t)$
which is indeed the form shown for $N(t)$ in the top line above.

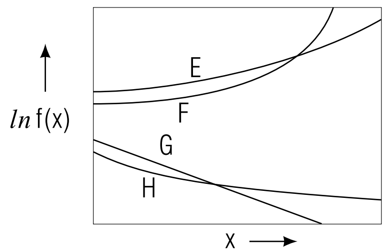
S-11 *(from Appendix A)*

$ln\,(AB) = ln(A) + ln(B)$; $ln\,[\exp(C)] = C$

MODEL EXAM

1. Which of the curves below could not correspond to $y(x)$ being a decaying exponential function of x? Justify your choice(s).





2. From the three observations reported below, show graphically that the data are in agreement with exponential decay and determine the mean life of the species.

t	$N(t)$
0 min.	639
2 min.	217
4 min.	73

3. Describe the relationship between the exponential decay law and experimental data for a finite number of decaying systems.

4. Derive the exponential decay law for the number of undecayed systems, starting from the "no aging" (constancy of decay constant) assumption.

Brief Answers:

1. A, B, C, E, F, H

2. 1.85 min, with a small experimental fluctuation.