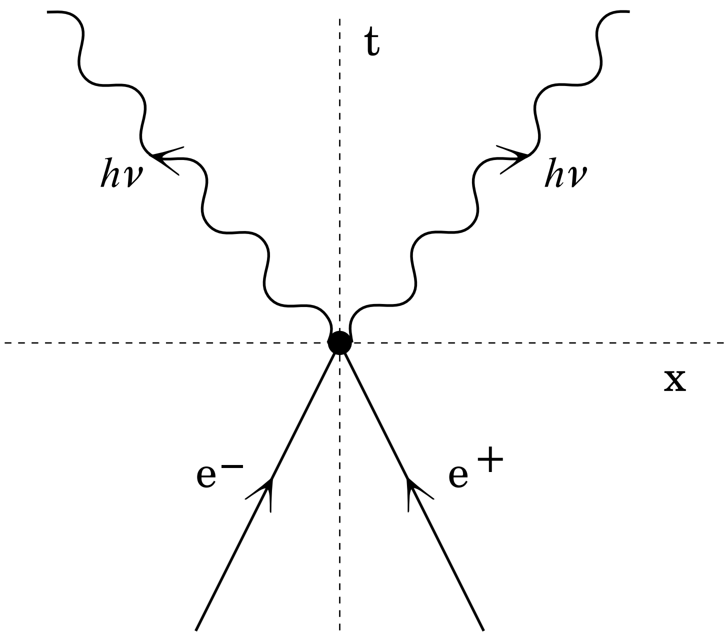
MISN-0-308

TOPICS IN RELATIVITY: DOPPLER SHIFT AND PAIR PRODUCTION



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

TOPICS IN RELATIVITY: DOPPLER SHIFT AND PAIR PRODUCTION

by
J. H. Hetherington

Title: **Topics in Relativity: Doppler Shift and Pair Production**

Author: J. H. Hetherington, Michigan State University

Version: 2/1/2000 Evaluation: Stage B0

Length: 1 hr; 16 pages

Input Skills:

1. Transform 4-vectors from one given Lorentz frame to another (MISN-0-75).
2. Transform energy and momentum from one Lorentz frame to another (MISN-0-75).

Output Skills (Knowledge):

K1. Derive the Doppler shift formula using the Lorentz transformation.

K2. State the minimum photon energy required to produce a pair of particles, each of mass M.

K3. Write down the energy-momentum 4-vector for a photon.

External Resources (Required):

1. R. T. Weidner and R. L. Sells, *Elementary Modern Physics*, 3rd ed., Allyn and Bacon, Boston (1980). For access, see this module's *Local Guide*..

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

PROJECT STAFF

Andrew Schnepp	Webmaster
Eugene Kales	Graphics
Peter Signell	Project Director

ADVISORY COMMITTEE

D. Alan Bromley	Yale University
E. Leonard Jossem	The Ohio State University
A. A. Strassenburg	S. U. N. Y., Stony Brook

Views expressed in a module are those of the module author(s) and are not necessarily those of other project participants.

© 2001, Peter Signell for Project PHYSNET, Physics-Astronomy Bldg., Mich. State Univ., E. Lansing, MI 48824; (517) 355-3784. For our liberal use policies see:

`http://www.physnet.org/home/modules/license.html`.

TOPICS IN RELATIVITY: DOPPLER SHIFT AND PAIR PRODUCTION

by

J. H. Hetherington

1. Particles with Zero Mass

Particles such as the photon which have zero rest mass still have their momentum and energy connected by the formula:

$$E^2 = m^2c^4 + p^2c^2 \tag{1}$$

but since $m = 0$ we have

$$E = |\vec{p}|c$$

or:

$$E/c = |\vec{p}|.$$

Therefore the 4-momentum vector for a photon is:

$$(\vec{p}, |\vec{p}|)\,.$$

We can therefore consider how a photon changes when observed in a moving coordinate system.

An important property of a photon is:

$$E = h\nu. \tag{2}$$

That is, the photon energy is proportional to frequency, and the proportionality constant is Planck's constant h.[1]

2. Doppler Shift

Imagine a photon moving in the $-x$ direction (relative to a frame of reference (S1) viewed by an observer (S2) moving in the $+x$ direction. In frame S1 we have for the photon:

$$(-h\nu/c, h\nu/c).$$

[1] $h = 6.625 \times 10^{-34}$ Joules sec

Therefore in frame S2 we have:

$$\begin{pmatrix} -h\nu'/c \\ h\nu'/c \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} -h\nu/c \\ h\nu/c \end{pmatrix} \tag{3}$$

or

$$-\frac{h'}{c} = -(1+\beta)\gamma\frac{h\nu}{c}.$$

i.e. Both equations implied by Eq.(3) are the same and:

$$\nu\,' = \frac{1+\beta}{\sqrt{1-\beta^2}}\,\nu. \tag{4}$$

This formula shows that an observer moving toward a photon source will see the photon as having higher frequency (blue shifted) than an observer stationary with respect to the source. This effect is frequently used in astronomy where moving stars or gases have spectral lines (say of hydrogen) which are shifted from those produced by experiments in the laboratory. From the size of this shift it is possible to determine the component of the velocity of a star along the line of sight.

▷ Work problem 2-29 of WSM[2]to make a coordinate space derivation of the Doppler effect.

3. Transverse Or 2nd Order Doppler Shift

An observer moving perpendicular to the motion of a photon will also notice a Doppler shift although it is proportional to β^2 and therefore is not generally observable except in unusual situations. This effect is derived as follows.

A photon is observed to arrive from a direction perpendicular to the motion of its source. Assume the source is moving in the x-direction and that the photon arrives moving in the −y-direction. (The photon in the frame of its own source is moving in a direction not perpendicular to the motion, however.) The transformation to a frame moving in the x-direction (source direction) with velocity $v = \beta c$ is given by the 4-dimensional (x, y, z, t) matrix (we will find that we need more than just

[2] R. T. Weidner and R. L. Sells, *Elementary Modern Physics*, 3rd ed., Allyn and Bacon, Boston (1980). For access, see this module's *Local Guide*.

x and t coordinates):[3]

$$\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}.$$

The 4-vector for our arriving photon moving in the $-y$-direction having frequency $\nu\,'$ is just:

$$\begin{pmatrix} 0 \\ -h\nu\,'/c \\ 0 \\ h\nu\,'/c \end{pmatrix}.$$

If the observer is moving in the x-direction, along the source, the photon will be observed traveling at an angle to the $-y$-direction. The 4-vector for the photon apparently leaving the source at some angle θ to the $-y$-direction is:

$$\begin{pmatrix} h\nu\sin\theta/c \\ -h\nu\cos\theta/c \\ 0 \\ h\nu/c \end{pmatrix}.$$

Multiplying out, one obtains 4 equations; the z equation is $0 = 0$, the others are:

$$\frac{\gamma h\nu}{c}\sin\theta - \beta\gamma\frac{h\nu}{c} = 0 \tag{5}$$

$$-\frac{h\nu}{c} = -\frac{h\nu\,'}{c} \tag{6}$$

$$-\beta\gamma\frac{h\nu}{c}\sin\theta + \gamma\frac{h\nu}{c} = \frac{h\nu\,'}{c}. \tag{7}$$

Eq. (5) gives directly $\sin\theta = \beta$ so that $\cos\theta = \sqrt{1-\beta^2}$. Then Eq. (6) gives $\nu\,' = \nu\sqrt{1-\beta^2}$ which is the transverse Doppler shift. It is easily shown that Eq. (7) is consistent with these results. Note that $\nu\,'$ is less than ν so that the transverse Doppler shift is to the red. Note that the condition that the motion be perpendicular to the direction of the arriving photon is important because the ordinary Doppler shift will be included if this perpendicularity is not rigidly maintained.

[3]See "Relativistic Space-Time: Four-Vectors" (MISN-0-75).

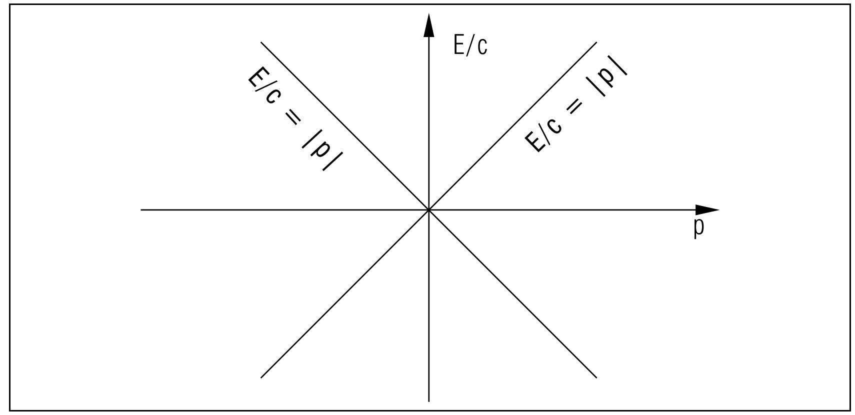


Figure 1.

4. Pair Production

A positron is the same as an electron except for the sign of its charge. A positron is an antiparticle to the electron. That is, a positron and an electron can come together and be annihilated. The rest energy has to be dissipated and generally two photons are produced as a result of pair annihilation. The inverse process, that of pair production,is considered here. A single photon cannot produce an electron-positron pair because of energy-momentum conservation.

First consider the possible components of the photon vector. If we plot the E/c and p components, since they are the same they must always lie on a 45° line (see Fig. 1).

Now an electron-positron pair's total energy and momentum can be plotted on the same graph. We know that in order for a single photon to produce an (e^-p^+) pair some possible (e^-p^+) momentum-energy must lie on one of the photon lines in Fig. 1. The total (e^-p^+) 4-momentum is

$$\begin{pmatrix} \vec{p}_e \\ E_e/c \end{pmatrix} + \begin{pmatrix} \vec{p}_p \\ E_p/c \end{pmatrix} = \begin{pmatrix} \vec{p}_e + \vec{p}_p \\ E_e/c + E_p/c \end{pmatrix} = \begin{pmatrix} \vec{p} \\ E/c \end{pmatrix}$$

where $E_n/c = \sqrt{(m_n c)^2 + \vec{p}_n^{\,2}}$ and $\vec{p}_n$ and m_n are the appropriate particle's momentum ($\vec{p}_e$, $\vec{p}_p$, or $\vec{p}$) and mass ($m_e = m_p = m$ and $2m$). Now it is possible to show that for a total momentum $\vec{p}$:

$$E/c \geq \sqrt{(2mc)^2 + (\vec{p})^2} \tag{8}$$

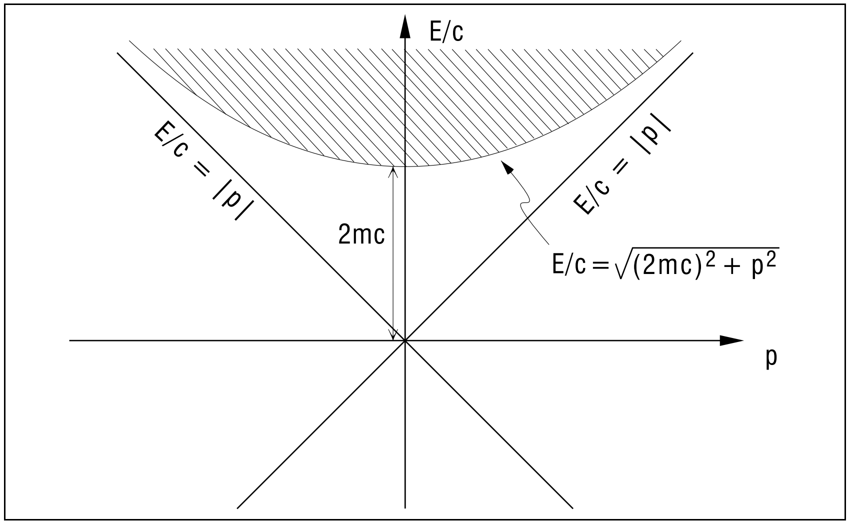


Figure 2.

To prove that Eq. (8) is true, first define $\vec{p} = \vec{p}_e + \vec{p}_p$ as above and $\vec{Q} = \vec{p}_e - \vec{p}_p$, then express $E/c + \sqrt{(mc)^2 + (p_e)^2} + \sqrt{(mc)^2 + (p_p)^2}$ in terms of $\vec{P}$ and $\vec{Q}$ and show that their expression is a minimum when $Q = 0$ and that in this case the equality in Eq. (8) holds.

Figure 2 shows the region in the $(E - p)$ plane where it is possible for an electron-positron pair to exist. Their region does not overlap the line $(E/c = |\gamma|)$ but is only asymptotic to it. Therefore a single photon cannot produce an electron positron pair.

However, if a photon comes near a heavy nucleus some momentum can be transferred to the nucleus with very little energy transfer.

▷ Show that a stationary proton that absorbs momentum $(2m_ec)$ absorbs only a small amount of energy compared to $(2m_ec)$. The proton mass is $1836\, m_e$.

Transfer of momentum with little energy transfer is a nearly horizontal vector in the $(p, E/c)$ plane. Such a vector added to the photon 4-vector can easily make the initial 4-momentum overlap the final 4-momentum.

The minimum energy the photon can have and still produce an e-p pair is connected by a horizontal line in Fig. 2 to the minimum of the e-p

energy-momentum domain. Therefore $h\nu \geq 2m_ec^2$.

5. Pair Annihilation

In WSM[4] study Section 4-4 through the second paragraph on page 120, plus Section 4-5, especially the subsection on pair annihilation.

[4]R. T. Weidner and R. L. Sells, *Elementary Modern Physics*, 3rd ed., Allyn and Bacon, Boston (1980). For access, see this module's *Local Guide*.

LOCAL GUIDE

The readings for this unit are on reserve for you in the Physics-Astronomy Library, Room 230 in the Physics-Astronomy Building. Ask for them as "The readings for CBI Unit 308." Do **not** ask for them by book title.

PROBLEM SUPPLEMENT

Note: Problems 7 and 8 also occur in this module's *Model Exam*.

1. A photon with energy $h\nu$ is emitted by an initially stationary atom of mass M. What is the recoil velocity of that atom?
2. A 1.0 MeV/c momentum positron collides with a stationary electron and they annihilate to give two photons, one going forward (along the initial positron direction) and one backward. What are the energies of the two photons?
3. A photon is moving along the y-axis according to a stationary (S1) observer. In a frame moving along the x-axis at velocity v (S2) what is the direction of the photon?
4. What is the energy of a photon resulting from annihilation of a positron-electron pair at rest?
5. What is the minimum energy of a photon that can cause pair production?
6. A spectral line normally at $\lambda = 4000$ angstroms in the laboratory is seen to be at $\lambda = 5000$ angstroms in the spectrum of a distant galaxy. At what velocity is the galaxy moving relative to earth? Is it moving toward or away from earth?
7. A photon, energy $h\nu$, moving in the $+x$ direction is observed from a frame moving in the minus x direction. How fast must the frame be moving if the photon energy is observed to have twice its original energy?
8. A positron and an electron collide and annihilate. The positron is moving at $0.60c$, the electron at $0.80c$ (both in the $+x$ direction).
   a. In what frame are they moving in equal and opposite directions?
   b. What are the photon energies in the original frame?

Brief Answers:

1. Initial atom 4-vector: $\begin{pmatrix} 0 \\ Mc \end{pmatrix}$

   Final atom and photon 4-vectors:

$$\begin{pmatrix} p \\ \sqrt{(M'c)^2 + p^2} \end{pmatrix} + \begin{pmatrix} h\nu/c \\ h\nu/c \end{pmatrix} = \begin{pmatrix} 0 \\ Mc \end{pmatrix}$$

   Note that M' is less than M because some of the atom's energy is converted into mass. For most real atoms $M' \approx M$. Therefore:

$$p + h\nu/c = 0 \qquad \text{or} \qquad p = -\frac{h\nu}{c}$$

   From momentum alone we can derive velocity:

$$\frac{p}{c} = \frac{M(\nu/c)}{\sqrt{1-\beta^2}} = \frac{h\nu}{c^2}$$

$$\frac{\beta}{\sqrt{1-\beta^2}} = \frac{h\nu}{Mc^2} \equiv \alpha$$

$$\beta^2 = \alpha^2(1-\beta^2)$$

$$(1+\alpha^2)\beta^2 = \alpha^2$$

$$\beta^2 = \frac{\alpha^2}{1+\alpha^2}$$

$$\beta = \frac{\alpha}{\sqrt{1+\alpha^2}}$$

   If $(h\nu/Mc^2) \ll 1$ then: $\beta \approx \alpha = \dfrac{h\nu}{M'c^2}$

   or

$$\beta = \frac{pc}{E} = \frac{p}{\sqrt{(M'c)^2 + p^2}}$$

   If $(p/Mc) = (hc/Mc^2) \ll 1$, then: $\beta = \dfrac{h\nu}{M'c^2}$

2. $\begin{pmatrix} p \\ \sqrt{m^2c^2 + p^2} \end{pmatrix} + \begin{pmatrix} 0 \\ mc \end{pmatrix} = \begin{pmatrix} -h\nu'/c \\ h\nu'/c \end{pmatrix} + \begin{pmatrix} h\nu/c \\ h\nu/c \end{pmatrix}$

$$p = \frac{h\nu}{c} - \frac{h\nu'}{c}$$

$$mc + \sqrt{m^2c^2 + p^2} = \frac{h\nu}{c} + \frac{h\nu'}{c}$$

   Hence:

$$\frac{h\nu}{c} = \frac{1}{2}(mc + \sqrt{(mc)^2 + p^2} + p)$$

$$\frac{h\nu'}{c} = \frac{1}{2}(mc + \sqrt{(mc)^2 + p^2} - p)$$

$$h\nu' = \frac{1}{2}(0.51 + \sqrt{(0.51)^2 + (1.0)^2} \pm 1.0)\,\text{MeV} = \frac{1}{2}(1.63 \pm 1.0)\,\text{MeV}$$

$$h\nu = 1.316\,\text{MeV}$$

$$h\nu' = 0.316\,\text{MeV}$$

3. $\begin{pmatrix} p_x \\ p_y \\ E \end{pmatrix} = \begin{pmatrix} \gamma & 0 & -\beta\gamma \\ 0 & 1 & 0 \\ -\beta\gamma & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} 0 \\ h\nu/c \\ h\nu/c \end{pmatrix} = \begin{pmatrix} -\beta\gamma h\nu/c \\ h\nu/c \\ \gamma h\nu/c \end{pmatrix}$

   Check: $(\gamma h\nu/c)^2 = (h\nu/c)^2 + (\beta\gamma h\nu/c)^2$.

   Therefore: $1/(1-\beta^2) = 1 + \beta^2/(1-\beta^2)$

   which is an identity: prove that it is!

   $\sin\theta = -\beta\gamma/\gamma$ if θ = angle from y-axis toward x-axis hence $\sin\theta = -\beta$.

4. 0.51 MeV

5. 1.02 MeV

6 $\dfrac{c}{\lambda'} = \dfrac{1+\beta}{\sqrt{1-\beta^2}} \cdot \dfrac{c}{\lambda}$

   or $\lambda\sqrt{1-\beta^2} = (1+\beta)\lambda'$

$$\frac{1+\beta}{\sqrt{1-\beta^2}} = \frac{\lambda}{\lambda'} \equiv \alpha$$

$$1 + 2\beta + \beta^2 = (1+\beta)^2 = \alpha^2(1-\beta^2)$$

$$\beta^2(1+\alpha^2) + 2\beta + (1-\alpha^2) = 0$$

$$\beta = \frac{-2 \pm \sqrt{4 - 4(1-\alpha^4)}}{2(1+\alpha^2)} = \frac{-1 \pm \sqrt{1 - 1 + \alpha^4}}{1+\alpha^2}$$

$$\beta = \frac{-1 \pm \alpha^2}{1+\alpha^2} \rightarrow \beta = \frac{\alpha^2 - 1}{\alpha^2 + 1}$$

$$\alpha = \frac{4}{5} = 0.80$$

$$\beta = \frac{0.64 - 1}{1.64} = 0.2195$$

7. $\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} h\nu/c \\ h\nu/c \end{pmatrix} = \begin{pmatrix} \gamma(1+\beta)h\nu/c \\ \gamma(1+\beta)h\nu/c \end{pmatrix}$

   Therefore: $\gamma(1+\beta) = 2$

   $(1+\beta)^2 = 4(1-\beta^2)$

   Method 1: Expand the above to get this quadratic equation:

   $5\beta^2 + 2\beta - 3 = 0$

   $\beta = \dfrac{-2 \pm \sqrt{4 + 4(3\cdot 5)}}{10} = \dfrac{-2 \pm 2\cdot 4}{10} = 0.6$ hence $v = 0.6c$.

   Method 2: Factor the β equation :

   $1 + \beta = 4(1-\beta)$

   $5\beta = 3$

   $\beta = 0.6$

8. The two 4-vectors are: $\begin{pmatrix} m\beta_1\gamma_1 c \\ m\gamma_1 c \end{pmatrix}$ and: $\begin{pmatrix} m\beta_2\gamma_2 c \\ m\gamma_2 c \end{pmatrix}$

   a. $\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} mc\beta_1\gamma_1 + mc\beta_2\gamma_2 \\ mc\gamma_1 + mc\gamma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ ? \end{pmatrix}$

   $mc\left[\gamma(\beta_1\gamma_1 + \beta_2\gamma_2) - \beta\gamma(\gamma_1 + \gamma_2)\right] = 0$

   $\beta = \dfrac{\beta_1\gamma_1 + \beta_2\gamma_2}{\gamma_1 + \gamma_2}$

   But: $\beta_1 = 0.6$ so $\gamma_1 = 1/0.8$ and $\beta_2 = 0.8$ so $\gamma_2 = 1/0.6$

   Therefore: $\beta = \dfrac{(.6/.8) + (.8/.6)}{(1/.8) + (1/.6)} = 1/1.4$

   b. $\begin{pmatrix} h\nu_1/c - h\nu_2/c \\ h\nu_1/c + h\nu_2/c \end{pmatrix} = \begin{pmatrix} mc(\beta_1\gamma_1 + \beta_2\gamma_2) \\ mc(\gamma_1 + \gamma_2) \end{pmatrix}$

   Therefore:

   $\dfrac{h\nu_1}{c} = \dfrac{mc}{2}(\beta_1\gamma_1 + \beta_2\gamma_2 + \gamma_1 + \gamma_2)$

   $\dfrac{h\nu_2}{c} = \dfrac{mc}{2}(\gamma_1 + \gamma_2 - \beta_1\gamma_1 - \beta_2\gamma_2)$

   $h\nu_1 = \dfrac{mc^2}{2}\left(\dfrac{.8}{.6} + \dfrac{.6}{.8} + \dfrac{1}{.6} + \dfrac{1}{.8}\right) = 2.5\, mc^2$

   $h\nu_2 = \dfrac{mc^2}{2}\left(\dfrac{5}{3} + \dfrac{5}{4} - \dfrac{4}{3} - \dfrac{3}{4}\right) = \dfrac{2.5}{6}\, mc^2$

MODEL EXAM

1. See Output Skills K1-K3 in this module's *ID Sheet*. One or more of these skills, or none, may be on the actual exam.

2. A photon, energy $h\nu$, moving in the $+x$ direction is observed from a frame moving in the minus x direction. How fast must the frame be moving if the photon energy is observed to have twice its original energy?

3. A positron and an electron collide and annihilate. The positron is moving at $0.60c$, the electron at $0.80c$ (both in the $+x$ direction).

   a. In what frame are they moving in equal and opposite directions?

   b. What are the photon energies in the original frame?

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 7.

3. See this module's *Problem Supplement*, problem 8.