PHYSNET

MISN-0-301

MATRIX ALGEBRA

$$\begin{pmatrix} g & 0 & +\beta\gamma \\ 0 & 1 & 0 \\ +\beta\gamma & 0 & g \end{pmatrix} \cdot \begin{pmatrix} 0 \\ p \\ (m^2c^2 + p^2)^{\frac{1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} +\beta\gamma\,(m^2c^2 + p^2)^{\frac{1}{2}} \\ p \\ \gamma\,(m^2c^2 + p^2)^{\frac{1}{2}} \end{pmatrix}$$

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

MATRIX ALGEBRA

by
J. H. Hetherington
Michigan State University

Title: **Matrix Algebra**

Author: J. H. Hetherington, Dept. of Physics, Michigan State Univ., E. Lansing, MI 48824

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Input Skills:

1. Solve a system of simultaneous linear equations (MISN-0-401).

Output Skills (Knowledge):

K1. Vocabulary: matrix, determinant, inverse of a matrix.

Output Skills (Rule Application):

R1. Add or multiply given matrices.

R2. Determine whether two given matrices are inverses of each other.

R3. Evaluate the determinant of a given 2×2 or 3×3 matrix.

R4. Write a given system of linear equations in matrix form.

External Resources (Required):

1. *The Mathematics of Physics and Chemistry* by H. Margenau and G. M. Murphy, Second Edition (Van Nostrand: New York, 1956). For availability, see this module's *Local Guide*.

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OF PROJECT PHYSNET

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Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

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1. Introduction

Matrix Algebra is an extremely compact and powerful way of doing much of the mathematics in physics. Almost all of the physics which uses matrix algebra could be done without matrix algebra—but matrix algebra is a much more compact way of doing things. With matrix algebra the theory of linear equations and vector algebra are both combined into one subject.

2. Resources

Study the following readings. The excerpts are from *The Mathematics of Physics and Chemistry* by H. Margenau and G. M. Murphy, Second Edition (Van Nostrand: New York, 1956), hereafter designated MM. For availability, see this module's *Local Guide*.

Skill:	**Readings**:
K1a	Sections 10.0, 10.1
K1b	Sections 10.2
K1c	Sections 10.3, 10.5, and 10.7 to "A = diag(A_1,A_2...)" "The transposed matrix..." to "The rule holds for any number of factors."
R1	Section 10.6, up to eq. 10-7
R2	Section 10.7
R3	Section 10.3
R4	Section 10.8, up to eq. 10-12, and Section 10.9 .

3. Matrix Manipulation

Note that matrix multiplication as defined in eq. 10-6 of MM is equivalent to this scheme:

Multiply columns of the second matrix by rows of the first matrix. [The product is the first element of the row times the first element of the column plus the second element of the row times the second element of the column, and so on.] The results is placed in the row and column of the resultant matrix corresponding to the number of the row of the first matrix and the number of the column of the second matrix. We illustrate this procedure:

1. Take first row of A and first column of B

$$\begin{pmatrix} A_{11} & A_{12} & \dots \\ A_{21} & A_{22} & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \dots \\ B_{21} & B_{22} & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \end{pmatrix}$$

first row of A: $\begin{pmatrix} A_{11} & A_{12} & \dots \end{pmatrix}$

first column of B: $\begin{pmatrix} B_{11} \\ B_{21} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$

2. Multiply row and column to obtain a single number:

$C_{11} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{21} + \ldots + A_{1N}B_{N1}$

3. Place this number in the first row and column of C:

$$\begin{pmatrix} C_{11} & \cdot & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \end{pmatrix}$$

4. Take first row of A and second column of B:

$$\begin{pmatrix} A_{11} & A_{12} & \dots \\ A_{21} & A_{22} & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \dots \\ B_{21} & B_{22} & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \\ \cdot & \cdot & \dots \end{pmatrix}$$

5. Multiply and obtain:

   $C_{12} = A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{22} + \ldots + A_{1N}B_{N2}$

6. Place in first row and second column of C:

$$\begin{pmatrix} C_{11} & C_{12} & \ldots \\ \cdot & \cdot & \ldots \\ \cdot & \cdot & \ldots \\ \cdot & \cdot & \ldots \\ \cdot & \cdot & \ldots \end{pmatrix}$$

7. Continue in this fashion until all the elements of the product matrix have been determined.

With practice you can easily do this mentally for small (2×2 and 3×3) matrices.

4. 2×2 and 3×3 Determinants

a. Caution: This method does not generalize to larger determinants.

(i) 2×2: Take the product on the principle diagonal and subtract the product on the cross diagonal:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

(ii) 3×3: Take the product on the principle diagonal and subtract the product on the cross diagonal:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= aei + bfg + dhc - ceg - bdi - hfa$$

The arrows pass through factors of the various terms.

b. Laplace's expansion applied to a 3×3 determinant:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - gf) + c(dh - ge).$$

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LOCAL GUIDE

The readings for this unit are on reserve for you in the Physics-Astronomy Library, Room 230 in the Physics-Astronomy Building. Ask for them as "The readings for CBI Unit 301." Do **not** ask for them by book title.

PROBLEM SUPPLEMENT

1. a. Multiply: $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = ?$

   b. Add: $\begin{pmatrix} a & b \\ -b & d \end{pmatrix} + \begin{pmatrix} a & 2b \\ b & 0 \end{pmatrix} = ?$

2. Show that $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1}$

3. a. Find the determinant of: $\begin{vmatrix} a & b \\ b & a \end{vmatrix}$

   b. Find the determinant of:
   $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{vmatrix}$$

4. Multiply:

   a. $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} a \\ 2a \end{pmatrix}$

   b. $\begin{pmatrix} a & 2a \end{pmatrix}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$

   c. $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

   d. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}$

5. Use Problem 2 to solve these linear coupled equations for x and y:
   $$(\cos\theta)x + (\sin\theta)y = 1$$
   $$(-\sin\theta)x + (\cos\theta)y = 2$$
   Use the matrix inverse method of Sec. 10.9, Margenau and Murphy.

6. Special Matrices: You should know, without actually multiplying, the result of multiplying:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \text{any matrix } A \\ \text{any column matrix} \end{pmatrix}$$

However, for practice right now, carry out the multiplication explicitly.

Brief Answers:

1. a. $\begin{pmatrix} 5 & 10 \\ 3 & 12 \end{pmatrix}$

   b. $\begin{pmatrix} 2a & 3b \\ 0 & d \end{pmatrix}$

2. Multiply and get unit matrix:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3. a. $a^2 - b^2$

   b. $(1)(5)(1) + (2)(6)(1) + (3)(4)(1) - (3)(5)(1) - (4)(2)(1) - (1)(1)(6) = 5 + 12 + 12 - 15 - 8 - 6 = 0$

4. a. a column matrix: $\begin{pmatrix} a(a+2b) \\ a(b-2a) \end{pmatrix}$

   b. a row matrix: $\begin{pmatrix} a(a+2b) & a(b-2a) \end{pmatrix}$

   c. a 1×1 matrix, a scalar: 4

   d. a 2×2 matrix: $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

5. The problem can be written: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   Multiply by inverse (which we know from Problem 2):

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -2\sin\theta \\ \sin\theta & +2\cos\theta \end{pmatrix}$$

   $x = \cos\theta - 2\sin\theta$; $\quad y = \sin\theta + 2\cos\theta$

6. Same matrix A or same column matrix.

MODEL EXAM

1. See Output Skill K1 in this module's *ID Sheet*.

2. Prove that these two matrices are inverses:

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

3. Find the determinants of: (a): $\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}$ and (b): $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 3 & 1 \end{vmatrix}$

4. Evaluate: $(A+B)C$ where:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}; \; B = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}; \; C = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

5. Using information from Problem 2, solve the equations:

   $(1/2)\,x - (1/2)\,y = a$

   $(1/2)\,x + (1/2)\,y = b$

Brief Answers:

2. Multiply and get: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. a. -4

   b. $5 + 36 + 36 - 8 - 18 - 45 = 77 - 71 = 6$

4. $$= \begin{pmatrix} 5 & 4 \\ 5 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 21 & 22 \\ 25 & 25 \end{pmatrix}$$

5. $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ -a+b \end{pmatrix}$$

   $x = a + b$

   $y = -a + b$