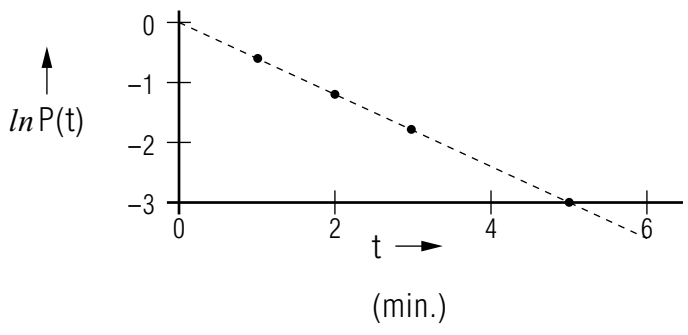
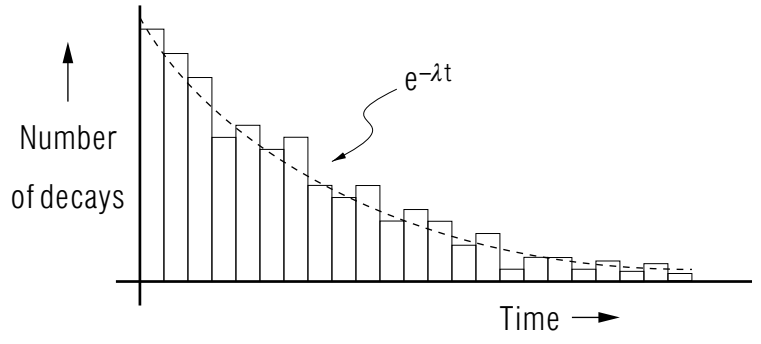


# EXPONENTIAL DECAY



## EXPONENTIAL DECAY

by  
Peter Signell

- 1. Nuclear Decay: Exponential**
  - a. The Exponential Decay Law (EDL) ..... 1
  - b. Decay-Constant and Mean-Life Values ..... 2
  - c. Half-Life ..... 2
  - d. Rate Equation ..... 2
- 2. Comparison of Edl to Experiment**
  - a. Results of One Experiment ..... 3
  - b. Repetitions of the Same Experiment ..... 3
  - c. The Number of Systems Involved ..... 4
- 3. The Probabilistic Edl**
  - a. Probability Applied to Decay ..... 4
  - b. The Probabilistic EDL ..... 4
  - c. Constant  $\lambda$ : No Internal Clock ..... 5
  - d. The “No-Aging” Assumption and QM ..... 5
- 4. Deducing Mean Life From Data**
  - a. The Semi-Log Plot ..... 6
  - b. Mean Life from Number Data ..... 6
  - c. Mean Life from Rate Data ..... 6
  - d. Dealing With Data Scatter ..... 7
- Acknowledgments** ..... 7
- A: Semi-Log Plots and Exponentials** ..... 7
- B: Relation of Mean and Half Lives** ..... 8

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Author: Peter Signell, Michigan State University

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**Input Skills:**

1. Vocabulary: natural logarithm, exponential function (MISN-0-401), probability.
2. Given a graphed straight line, determine slope, intercept, and equation.

**Output Skills (Knowledge):**

- K1. Vocabulary: mean life, half-life.
- K2. Describe the relationship between the exponential decay law and typical finite-number data.

**Output Skills (Rule Application):**

- R1. Determine whether or not a given set of decay data is consistent with an exponential description.
- R2. Determine the mean life for a given set of exponential decay data.

**External Resources (Required):**

1. A source of natural logarithms (table or calculator).
2. Several sheets of graph paper.

**Post-Options:**

1. "Quantum Tunnelling Through a Barrier: Pictures, Probability Flow" (MISN-0-265).

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## EXPONENTIAL DECAY

by  
Peter Signell

### 1. Nuclear Decay: Exponential

**1a. The Exponential Decay Law (EDL).** Suppose we examine identically prepared systems belonging to a single nuclear species which decays radioactively. One of the first experiments which comes to mind is that of measuring the times at which the decays occur in the sample at hand. When the decays have finally ceased, one can count the decays which were recorded and thereby learn the original number of radioactive systems which were undecayed at the beginning of observation. Then by subtraction one can find the number of such undecayed systems left at any particular time during the observations, and this number can be plotted as a function of time. Tens of thousands of such experiments have been performed on a great variety of decaying species with varying numbers of initial systems, and these experiments have led to the Exponential Decay Law, hereafter abbreviated EDL.<sup>1</sup> With few exceptions, the data on radioactive decays appear to be in agreement with this statement of the EDL: The number of systems still undecayed at time  $t$ , designated  $N(t)$ , is a decaying exponential function of time:<sup>2</sup>(see Fig. 1)

$$N(t) = N(0) e^{-\lambda t}. \quad \text{Question: [Q-1]} \quad (1)$$

The “decay constant,”  $\lambda$ , is uninfluenced by environmental characteristics such as temperature and pressure:<sup>3</sup> it is a characteristic number for each

<sup>1</sup>Not a widely used abbreviation.

<sup>2</sup> Question: [Q-1] refers to Question 1 in this module’s *Problem Supplement*.

<sup>3</sup>Except for one rare type known as “K-capture”.

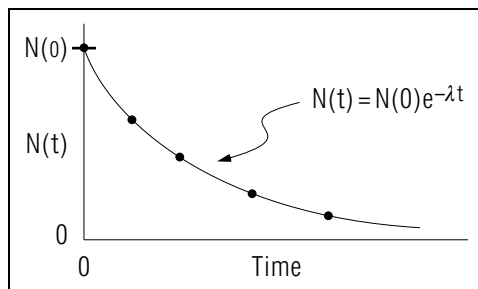


Figure 1. .

nuclear species.

**1b. Decay-Constant and Mean-Life Values.** Values of the decay constant range from 1 decay in  $10^{12}$  years for the alpha decay of  $\text{Sm}^{152}$  to 1 per  $10^{-23}$  second for the strong decay of the rho meson. The number of undecayed systems at time zero is usually under the control of the experimenter. For an exponentially-decaying species, its mean life  $\bar{t}$  can be shown, using calculus, to be the inverse of its decay constant:<sup>4</sup>

$$\bar{t} = 1/\lambda. \quad (2)$$

**1c. Half-Life.** The time it takes for half a sample to decay is called the sample’s “half life,”  $t_{1/2}$ :

$$N(t_{1/2}) = (1/2) N(0). \quad (3)$$

Then we can relate the half and mean lives (See Appendix B for details):

$$t_{1/2} = \bar{t} \cdot \ln 2, \quad (4)$$

where  $\bar{t}$  is the system’s mean life. Half life is often quoted in scientific literature because it is easily visualized and measured. It can be applied at any time  $t_1$ , thereby giving the time when half the  $t_1$  sample will have decayed. Substitution of Eq. (4) into Eq. (1) yields the interesting relation:

*Help: [S-5]*<sup>5</sup>

$$N(t) = N(0) \times \left(\frac{1}{2}\right)^{t/t_{1/2}}.$$

This says that after three half lives there is only one eighth of the sample remaining undecayed.

Note that one does not have to begin time zero when the sample is created: no matter when one examines a sample and simultaneously starts time running, at one half life later just half the observed sample will remain.

**1d. Rate Equation.** For exponential decay, the rate of decay,  $R(t)$ , obeys an equation similar to that for the number of undecayed systems:

$$R(t) = R(0) e^{-\lambda t}, \quad (5)$$

where  $\lambda$  is the same decay constant that occurs in the number equation, Eq. (1). The rate  $R(t)$ , is usually quoted in units of “number of disintegrations per unit time.” Here are three examples:

<sup>4</sup>See Appendix B for details.

<sup>5</sup>For help, see [S-5] in this module’s *Special Assistance Supplement*.

6.12 disintegrations per second

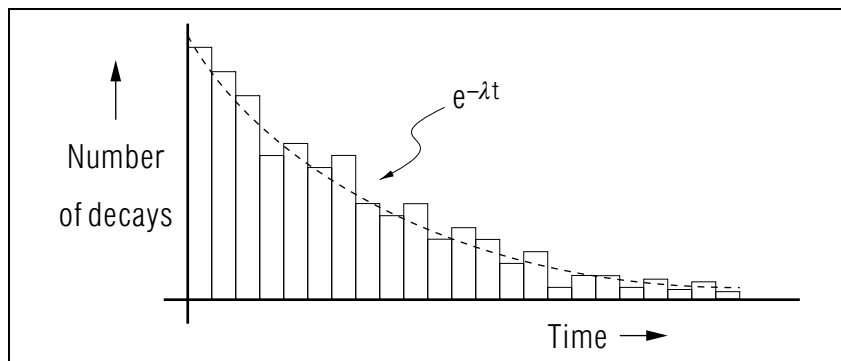
186 dis./min

$1.562 \times 10^{-9}$  dis./yr.

## 2. Comparison of Edl to Experiment

**2a. Results of One Experiment.** The exponential number and rate equations do not really coincide with what one usually finds experimentally. A typical set of experimental observations is shown in Fig. 2, where each vertical bar represents the total number of decays counted in the corresponding time interval or bin. Note that, far from obeying the monotonically decreasing exponential decay function of the EDL, some of the later bins have more counts than earlier bins. Furthermore if one took a complete second set of observations with the same original number of identically prepared systems, the second-set data would not reproduce those of the first set. The tops of the bars for the second set would still fluctuate about the same smooth exponential curve but the fluctuations would be unpredictably different.

**2b. Repetitions of the Same Experiment.** If one repeated the Fig. 2 experiment many times and kept adding the new observations into the corresponding time bins, the vertical scale in Fig. 2 would have to be continually changed in order to keep the exponential curve in the same place on the graph paper. As the number of counts in each bin increased, the fluctuating appearance of the tops of the bins would decrease and the



**Figure 2.** Typical results of decay measurements for a set of radioactive systems. The fluctuations are not due to experimental error.

centers of the tops would appear to approach the smooth curve. Of course the fluctuations would still be there but they would become very small on the scale of the graph.

**2c. The Number of Systems Involved.** It is now obvious that one must always write next to the exponential rate and number equations; “valid only as  $N(t) \rightarrow \infty$ .” However, it is possible to rewrite the equations and the basic assumption so as to be valid for any number of systems, even for a single decaying system. To do so, however, we have to change to a probabilistic description. *Question: [Q-3]*

## 3. The Probabilistic Edl

**3a. Probability Applied to Decay.** The problem is to cast the EDL and its rate equation into a form which is valid for a finite number of systems, even for a single decaying system. The solution is to rewrite the EDL in terms of the probability  $P(t)$  that any single specific system will still be undecayed at time  $t$ . Since all the decaying systems are identically prepared, each will have an identical time-evolving probability of still being undecayed.

**3b. The Probabilistic EDL.** Statistically, the probability of a single system still being undecayed at time  $t$  is defined experimentally and theoretically as the limit of the fraction of undecayed systems at that time:

$$P(t) = \lim_{N \rightarrow \infty} \frac{N(t)}{N(0)}, \quad (6)$$

where  $N(0)$  represents the total number of systems, decayed and undecayed. Applying this to the number and number rate equations, we get the probabilistic exponential decay law (EDL):

$$P(t) = e^{-\lambda t}, \quad (7)$$

and its rate equation:

$$r(t) = \lambda e^{-\lambda t}. \quad (8)$$

Evaluating the law at time zero, we find:

$$P(0) = 1.00, \quad (9)$$

which says that there is a 100% probability that the system is undecayed at time zero. That time is, of course, the time of creation of the system.

Our new exact statement of the basic assumption is:

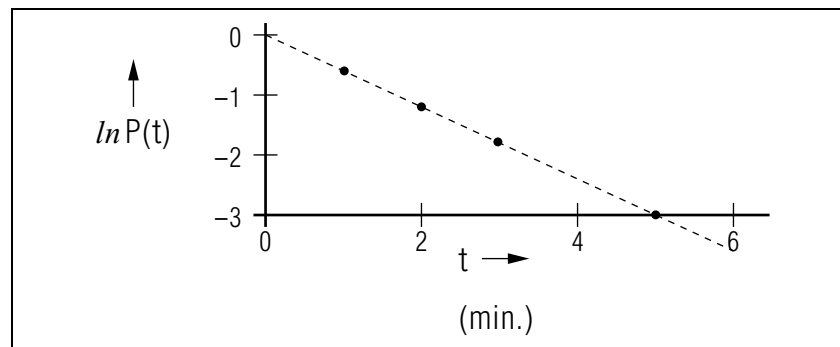
The rate of decay of the probability  $P(t)$  is, at any time, proportional to the undecay probability  $P(t)$  remaining at that time.

As an equation, this basic assumption is written:

$$r(t) = \lambda P(t). \quad (10)$$

**3c. Constant  $\lambda$ : No Internal Clock.** The basic EDL assumption, stated above, carries the implication that the proportionality constant  $\lambda$  is independent of time. This means that all times are the same to an undecayed system; that its character remains completely unchanged until the instant of its decay. Can you swallow that? Usually, if something is going to decay it must have an internal clock of some kind which runs down: it must “age” until that cataclysmic moment when the vital element stops. Without such an aging process, how can an EDL system decay at all? The derivation of the EDL gives no answer; for that one must look to Quantum Mechanics.

**3d. The “No-Aging” Assumption and QM.** The mechanism of radioactive decay is generally described by Quantum Mechanics, which views exponential decay as no more than a mathematical approximation to a certain segment in the life of a radioactive system. For example, the standard model of  $\alpha$ -decay uses Quantum Mechanics’ Time-Dependent Schrödinger Equation. This equation nicely produces a time-evolving probability for all except completely stable (non-decaying) systems. Thus for decaying (unstable) states it always violates the “no-aging” assumption of exponential decay summarized in Eq. (10). Nevertheless, the Time-Dependent Schrödinger Equation does produce the almost-exponential behavior observed for appropriate segments in the lives of appropriate systems.<sup>6</sup> Most radioactive systems of use in industry and basic research have enormous periods of time over which the EDL is an extremely accurate approximation and the “aging” is extremely small.<sup>7</sup>



**Figure 3.** Illustrative example of logarithmic decay as seen on a semi-log plot. The mean life is:  $\bar{t} = 1.67$  min.

## 4. Deducing Mean Life From Data

**4a. The Semi-Log Plot.** The first step in analyzing decay data is to make a semi-log plot. This means that a graph is made in which the vertical axis is effectively the logarithm of the dependent variable while the horizontal axis is the independent variable. This is accomplished by plotting each datum on a piece of semilog graph paper or, equivalently, by plotting the logarithm of each datum on linear graph paper as in Fig. 3. If the data fall on a straight line, the decay is exponential.<sup>8</sup>

**4b. Mean Life from Number Data.** For exponential decay data which are the number of undecayed systems at various times, the mean life is just the negative inverse of the semi-log plot’s slope:

$$\bar{t} = 1/\lambda; \quad \lambda \equiv -\text{slope}(\ln N \text{ vs. } t).$$

You can easily check the value given in the caption for Fig. 3, using these formulas. *Help: [S-3]*

**4c. Mean Life from Rate Data.** The mean life can be deduced from decay-rate data in a manner exactly parallel to that for number-of-undecayed-systems data. This is because the basic equation for the decay

<sup>6</sup>See “Quantum Tunneling Through a Barrier: Pictures of  $\alpha$ -Decay” (MISN-0-250) for a pictorial presentation of the three main segments in the lives of quasi-stable systems. The pictures and film were generated by solving the Time-Dependent Schrödinger Equation on a computer.

<sup>7</sup>To obtain a feeling for the conditions under which this happens, see “A Soluble Model of Radioactive Decay” (MISN-0-312, under construction).

<sup>8</sup>For a quick examination of the relationship between exponentiality and semi-log plots, see Appendix A, this module.

rate has exactly the same mathematical form as Eq. (1) for the number  $N(t)$ . Then the decay constant  $\lambda$  can be obtained immediately from a plot of  $[\ln R(t)]$  vs  $[t]$  and  $\bar{t}$  follows from  $\lambda$ .<sup>9</sup>

**4d. Dealing With Data Scatter.** Real-life data do not fall on a completely smooth curve but rather show a random-looking scatter about such a curve (see Fig. 2), and this must be dealt with in deducing a mean life. The scatter itself has two unrelated origins: experimental uncertainties or “errors,” and the ever-present quantum fluctuations. The effect of the former can usually be reduced through better and/or more costly experimental design, but the effect of quantum fluctuations can be reduced only by increasing the number of data. Methods for dealing with such data-scatter are presented elsewhere<sup>10</sup> but an understandable effect is that the deduced mean life value is made uncertain. Graphically speaking, the random character of the fluctuations make uncertain the location of the line on the semi-log plot. That uncertainty is transferred to the line’s slope and hence to the mean life. In many experimental situations, however, the data are so large in number that they make the  $\lambda$  and  $\bar{t}$  accurate enough for practical applications.

## Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

## A: Semi-Log Plots and Exponentials

An exponential decay function produces a straight line on a semi-log plot, and the slope of that line is the negative of the decay constant  $\lambda$  where, for example,

$$f(x) = f(0) \exp(-\lambda x).$$

<sup>9</sup>Practical applications almost always use rate data: see “Some Uses of Radioactivity” (MISN-0-252), in which standard units of radioactivity measurement are introduced and used.

<sup>10</sup>See “Linear Least Squares Fits to Data” (MISN-0-162, under construction).

To prove these relationships, we take the natural logarithm of the above equation, getting:<sup>11</sup> We now define a new dependent variable,

$$y(x) \equiv \ln f(x),$$

which linearizes the relationship between the new dependent and independent variables:

$$y(x) = y(0) - \lambda x.$$

Thus plotting  $(y)$  vs.  $(x)$  on a graph produces a straight line whose slope is  $(-\lambda)$ , the negative of the decay constant.

## B: Relation of Mean and Half Lives

We use these three equations from the main text of this module:

$$\bar{t} = 1/\lambda \tag{3}$$

$$N(t_{1/2}) = (1/2) N(0) \tag{4}$$

$$N(t) = N(0) e^{-\lambda t} \tag{2}$$

Follow these steps:

1. Evaluate the time in Eq. (2) at  $t_{1/2}$ , wherever  $t$  occurs.
2. In the result of Step 1, substitute Eq. (4) for  $N(t_{1/2})$  and then cancel the  $N(0)$  factors.
3. Take the natural logarithm of both sides of the result of Step 1 (follow the rules shown in Appendix A).
4. In the result of Step 3, substitute  $\lambda$  from Eq. (3) to get:

$$t_{1/2} = \bar{t} \cdot \ln 2,$$

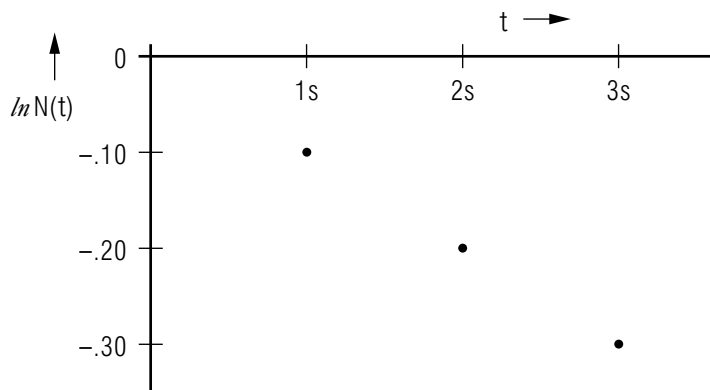
which is the desired result.

<sup>11</sup>Note:  $\ln(AB) = \ln(A) + \ln(B)$ ; and:  $\ln \exp(C) = C$ .

**PROBLEM SUPPLEMENT**

Note: problems 8 and 9 also occur on this module's *Model Exam*.

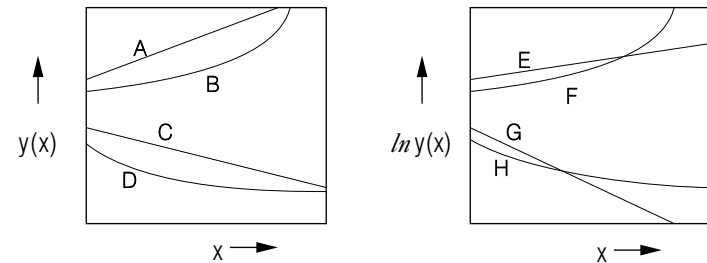
- By direct substitution of  $t = 0$  in Equation (1), determine and check the meaning of  $N(0)$ .
- Explain the correspondence between data and the exponential decay law, including time-bin fluctuations and the changes in their significance as the number of systems is increased.
- Show that Eq. (9) follows from the preceding equations.
- Verify Eq. (2).
- Sketch a curve, on a linear plot, which is an exponential function and a curve which is not. On a semilog plot, sketch a curve which represents an exponential function and one which does not.
- From the observations shown below, determine the mean life for the species being observed. *Help: [S-1]*



- From the three rate observations reported below, show graphically that the data are in agreement with exponential decay and determine the mean life of the species. *Help: [S-2]*

$t$ :	$R(t)$ :
0.00 sec	2638/sec
1.00 sec	755/sec
2.00 sec	216/sec

- Which of the curves below could not correspond to  $y(x)$  being a decaying exponential function of  $x$ ? Justify your choice(s).



- From the three observations reported below, show graphically that the data are in agreement with exponential decay and determine the mean life of the species.

$t$ :	$N(t)$ :
0.00 min.	639
2.00 min.	217
4.00 min.	73

**Brief Answers:**

- 10s.
- 0.799s; graph of  $\ln R(t)$  vs.  $t$  is a straight line.
- A, B, C, E, F, H.
- 1.85 min.

**SPECIAL ASSISTANCE SUPPLEMENT**

**MODEL EXAM**

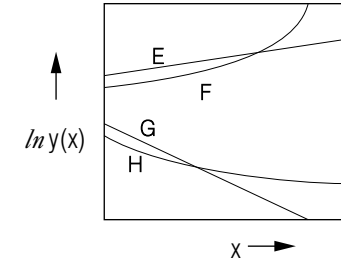
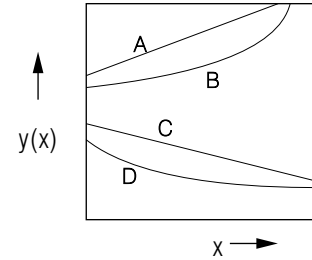
S-1 (from TX-Problem 6)  
 slope =  $\frac{\Delta \ln N}{\Delta t} = \frac{-0.10}{1 \text{ s}} = -\lambda$ ;  $\bar{t} = 1/\lambda = 10 \text{ s.}$

S-2 (from PS-Problem 7)  
 $\ln 2638 = 7.878$   
 $\ln 755 = 6.627$  > diff = -1.251  
 $\ln 216 = 5.375$  > diff = -1.252  
 slope =  $\frac{\Delta \ln R}{\Delta t} = \frac{-1.251}{1 \text{ s}} = -\lambda$ ;  $\bar{t} = 1/\lambda = 0.799 \text{ s.}$

S-3 (from TX-4b)  
 slope =  $\frac{\Delta \ln P}{\Delta t} = \frac{-3}{5 \text{ min}} = -\lambda$ ;  $\bar{t} = 1/\lambda = 1.67 \text{ min.}$

S-5 (from TX-1c)  
 From the text:  $N(t) = N(0)e^{-\lambda t}$ ;  $1/\lambda = \bar{t} = t_{1/2}/\ln 2$ .  
 Substituting one into the other:  
 $N(t) = N(0)e^{-t/(t_{1/2}\ln 2)} = N(0) [e^{-\ln 2}]^{t/t_{1/2}} = N(0) [1/2]^{t/t_{1/2}}.$

1. See Output Skills K1-K2 in this module's *ID Sheet*.
2. Which of the curves below could not correspond to  $y(x)$  being a decaying exponential function of  $x$ ? Justify your choice(s).



3. From the three observations reported below, show graphically that the data are in agreement with exponential decay and determine the mean life of the species.

$t$ :	$N(t)$ :
0.00 min.	639
2.00 min.	217
4.00 min.	73

**Brief Answers:**

1. See this module's *text*.
2. See this module's *Problem Supplement*, problem 8.
3. See this module's *Problem Supplement*, problem 9.