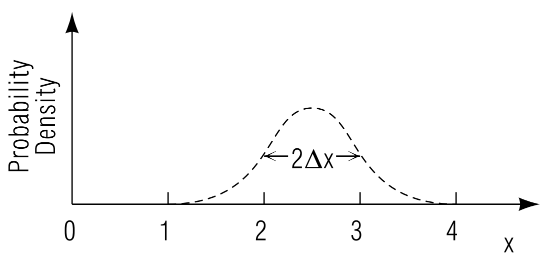
PHYSNET

MISN-0-241

THE UNCERTAINTY RELATIONS: DESCRIPTION, APPLICATIONS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

THE UNCERTAINTY RELATIONS:
DESCRIPTION, APPLICATIONS

by

Peter Signell

Title: **The Uncertainty Relations: Description, Applications**

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Input Skills:

1. Vocabulary: energy level, excited state, ground state (MISN-0-215); de Broglie wave, wave-particle duality, wave function (MISN-0-240).
2. Calculate the energy levels of a given atomic one-electron system (MISN-0-215).
3. Calculate the wavelength of spectral lines for transitions between given energy levels of atomic one-electron systems (MISN-0-215).
4. Determine the de Broglie wavelengths of particles of a given energy and determine the most probable scattering angles for diffraction by crystal lattices and single slits (MISN-0-240).

Output Skills (Knowledge):

K1. Vocabulary: level width, lifetime, uncertainty, radioactive, wave packet.

K2. State Heisenberg's uncertainty principle and illustrate it with at least two examples.

K3. State the energy-time uncertainty relation.

Output Skills (Problem Solving):

S1. Given a situation in which position and momentum (or lifetime and energy) are to be simultaneously measured, determine the possible precision of measurement of one variable given constraints on the precision of measurement of the other.

Post-Options:

1. "Wavefunctions, Probability, and Mean Values" (MISN-0-243).

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1. Introduction

1a. Waves Imply Uncertainties. The wave nature of a particle carries the implication that our knowledge of its momentum and position are somewhat uncertain (where is "the position" of a wave?). Utilizing the Schrödinger equation, we can prove Heisenberg's uncertainty principle: that the product of the position and momentum uncertainties has a universal minimum value.[1] Thus as a very accurate measurement of a particle's position dramatically decreases our uncertainty in its position, then the measurement simultaneously and uncontrollably increases the minimum possible uncertainty in the particle's momentum.

1b. Personal Reactions to Minimum Uncertainty. The relationship between position and momentum uncertainties is rather unique in physics. As one lower limit goes down the other goes up, and there is nothing whatever we can do about it. Some physicists dislike this effect; they feel that there should be no fundamental barrier to knowledge. Other physicists embrace the effect; they feel that there should be an inherent limit to knowledge so that our task is not endless.

1c. Experimental Evidence. No violation of the uncertainty principle has ever been found. Noting that all evidence had come from the physical universe, at one time it was postulated that perhaps the most basic life process, the replication of DNA, might violate the uncertainty principle. However, even for this process it was found that the principle was obeyed, albeit just barely.

2. The Two Uncertainty Relations

2a. Position-Momentum Relation. The position-momentum uncertainty relation, which can be precisely derived from the Schrödinger equation, states that if Δx is the uncertainty in the x-coordinate of the position of an object and Δp_x is the uncertainty in the x-component of

[1]See "Wave Functions, Mean Values and Probability" (MISN-0-243).

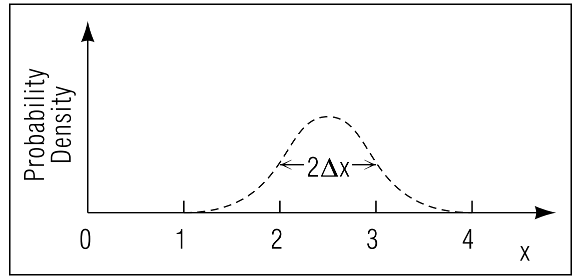

Figure 1. The probability density of a hypothetical particle, illustrating position uncertainty.

the momentum of that object, then:

$$\Delta x \Delta p_x \geq \hbar/2\,. \tag{1}$$

This is called the Heisenberg uncertainty relation.[2] Here is how you should read the equation: "delta x times delta p sub x is greater than or equal to h bar over 2." The quantity "h bar" is defined as "h divided by two pi": $\hbar = h/2\pi$.

2b. Energy-Time Relation. We can also show that for many situations

$$\Delta E \Delta t \approx \hbar\,, \tag{2}$$

which is sometimes referred to as an uncertainty relation for energy and time. It has recently been found that there does not appear to be any energy/time relation analogous to Eq. (1). Instead, we have this imprecise statement. Equation (2) is often a good approximation for processes on the scale of atoms.

2c. Definition of Uncertainty. The uncertainty of a measured quantity is defined as the root-mean-square (RMS) deviation from its mean value. Using the wave function and the associated probability density that describe a particular object, the uncertainty in the position of an object, Δx, can be shown to be half the "width" of the probability density function measured at half the "height" of the density function (this "dimension" is called the "half width at half height," or HWHH for short). Thus the "full width at half height" (FWHH) of the probability density is twice Δx. As an example, an object described by the probability density shown in Fig. 1 has a average position of 2.5, and an uncertainty in position of 0.5. For any other measured quantity, the probability density

[2]Eq. (1), above, is the precise general statement of the position-momentum uncertainty relation. Elementary textbooks sometimes state a relation which is only approximate and only developed for a particular case.

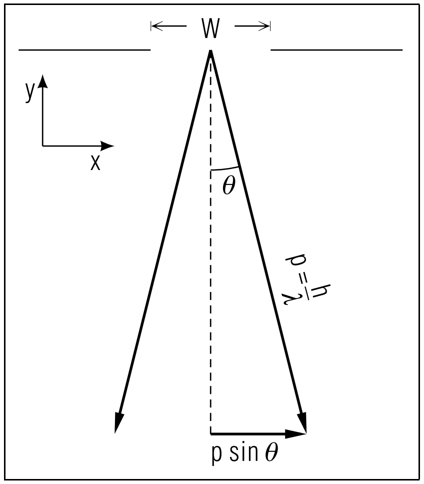

Figure 2. Photons diffracted by a slit have uncertainties $\Delta x = W/2$ and $\Delta p_x \approx p \sin\theta \approx h/W$.

is used to compute the mean value and RMS deviation of the variable. *Help: [S-2]* Thus given a complete mathematical description of the wave function of an object, we can compute Δx and Δp_x precisely. However, interesting and useful numbers can be obtained through rough estimation (e.g. when one knows that the particle has gone through a slit of width W, taking $\Delta x \approx W/2$).

3. The Position-Momentum Relation

3a. Photons Passing Through a Slit. As an illustration of the position-momentum uncertainty relation, consider photons incident upon a narrow slit. Let us estimate the uncertainty in the momentum and position of the photons after they have passed through the slit. If the width of the slit is W (see Fig. 2) and the wavelength of the incident light is λ, the first minima of the Fraunhofer diffraction pattern occurs at the angle θ where[3]

$$\sin\theta = \frac{\lambda}{W}\,. \tag{3}$$

There is a good probability that a photon will be scattered into this range, since it is most likely that a photon will end up within the central maximum of the diffraction pattern. The uncertainty in its x-component

[3]See "Fraunhofer Diffraction" (MISN-0-235).

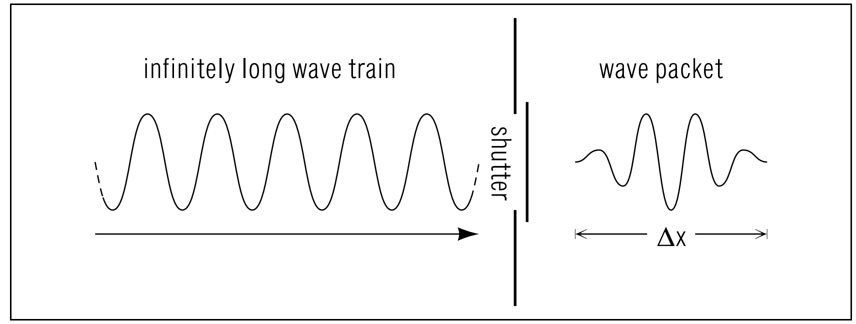

Figure 3. A wave packet is formed by opening and closing a light shutter.

of momentum after it has passed through the slit is thus (roughly)

$$\Delta p_x \approx p \sin\theta = \frac{h}{\lambda}\frac{\lambda}{W} = \frac{h}{W}\,, \tag{4}$$

where $p = h/\lambda$ is de Broglie's relation for the wavelength of a particle.[4] Since the uncertainty in the x-coordinate of the position of a photon that has just passed through the slit is $\Delta x \approx W/2$, and its uncertainty in p_x is h/W, then

$$\Delta p_x \Delta x \approx \frac{h}{2}\,. \tag{5}$$

Notice that this equation differs from Eq. (1) by a factor of 2π because we have not used the precise definition of uncertainty to calculate Δp_x.

3b. Light Passing Through a Shutter. A more revealing illustration of the relationship of the wave-particle duality of light to the uncertainty principle involves passing an infinitely long wave train through a light shutter. The shutter can be opened and closed to allow a short segment of the wave train, called a "wave packet," to pass through it (see Fig. 3). This wave packet may be used to represent a photon. In order to obtain a wave packet of length Δx, it is necessary to include a range of wave numbers Δk of the order

$$\Delta k \approx \frac{1}{\Delta x}\,. \tag{6}$$

But since $p = \hbar k$, this implies also a range of momenta $\Delta p_x = \hbar \Delta k$ and so

$$\Delta p_x \Delta x \approx \hbar\,. \tag{7}$$

[4]See "De Broglie Waves" (MISN-0-240).

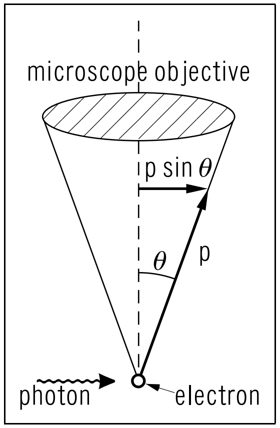

Figure 4. "Observing" an electron with a Heisenberg microscope.

It can be seen that in order to shorten the wave packet (and more precisely determine the position of the photon), Δk (and consequently Δp_x) must increase so that the product remains constant. Thus the uncertainty principle is intimately related to the wave-particle nature of matter and light.

3c. The Heisenberg Microscope. The Heisenberg microscope is a "gedanken," or thought, experiment which illustrates that the uncertainty principle can be applied to particles other than photons. Imagine that an electron is observed through a microscope where light is incident from the left, as shown in Fig. 4. If the electron is assumed to be initially stationary, its momentum is thus known exactly and we can then try to determine its position simultaneously. To observe where the electron is, one of the incident photons must strike the electron and be scattered into the microscope objective. However, when the photon bounces off the electron, it imparts momentum to the electron. Because the photon can enter the finite aperture of the microscope objective anywhere across its width, the x-component of the photon's momentum is not completely known, and the amount of momentum imparted to the electron is uncertain. The uncertainty in the x-component momentum of the electron after being struck is thus the same as the uncertainty in the x-component of the photon:

$$\Delta p_x = p \sin\theta = \frac{h}{\lambda}\sin\theta\,. \tag{8}$$

The resolving power, and thus the accuracy Δx with which the position of the electron can be determined, is

$$\Delta x = \frac{\lambda}{\sin\theta}\,, \tag{9}$$

where λ is the wavelength of the incident light.[5] Once again, the relationship between the uncertainties in position and momentum is found to be approximately

$$\Delta p_x \Delta x \approx h\,. \tag{10}$$

4. Lifetimes of Energy States

4a. Interpretation of ΔE and Δt. The energy-time uncertainty relation has a physical interpretation different from that of the position-momentum relation. In the latter, position and momentum have symmetrical roles; they can both be measured at a given time t. In the energy-time relation, however, ΔE and Δt play fundamentally different roles. ΔE is the uncertainty in a dynamical variable E, while Δt is the time interval during which the energy is uncertain by that amount. Thus, the situation is different because the energy is a function of time.

4b. Lifetimes and Energy Level Widths. An important application of $\Delta E \Delta t \approx \hbar$ is the lifetime-energy level width relation for radioactive systems. A "radioactive" system is one which is not stable, such as the excited state of an atom, a radioactive nucleus, or an unstable elementary particle. Because the system is not stable, it does not correspond just to one energy, but to a spread of energies ΔE, which is usually called the "level width." The average length of time a system remains in a certain state is known as the "lifetime" τ of the state. The lifetime τ is assumed to be of the same order of magnitude as Δt (its uncertainty). Consequently, the relationship between the lifetime and energy level width of a certain state is

$$\tau \Delta E \approx \hbar\,. \tag{11}$$

4c. Example: Excited States of an Atom. The lifetime-level width relationship can be applied to an excited state of an atom. An electron in an excited energy state will, after a certain length of time, undergo a spontaneous transition to another state of less energy. There is no way to

[5] See "Fraunhofer Diffraction" (MISN-0-235).

predict with certainty when the transition will occur. However we may predict the average lifetime of the excited state. Equation (11) tells us that the longer the lifetime, the smaller the uncertainty in the energy of the state. In a ground state, whose lifetime is infinite because it cannot undergo a spontaneous transition to a lower energy level, $\tau = \infty$. This gives $\Delta E = 0$, so the energies of ground states can be measured without any inherent uncertainty. The theory commonly used in theoretical chemistry and atomic and molecular physics predicts a single value also for each excited state of an atom or molecule. As can be seen from the above analysis, this cannot be true.

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **level width**: the spread of energies associated with an energy level of an excited state of an atom, molecule or nucleus.
- **lifetime**: the average length of time a radioactive system remains in a given state.
- **radioactive**: an adjective which describes a system which is unstable, and decays to another state by the spontaneous emission of radiation.
- **uncertainty**: the root-mean-square deviation of a quantity from its mean value.
- **wave packet**: a linear superposition of single-frequency, infinitely long wave trains, that has a finite length. The length of the wave packet depends inversely on the range of wavelengths of the single frequency waves that are included in the linear superposition.

PROBLEM SUPPLEMENT

$h = 6.63 \times 10^{-34}$ J s$= 4.14 \times 10^{-15}$ eV s

$c = 3.0 \times 10^{8}$ m/s

$\hbar c = hc/2\pi = 197.3 \times 10^{-9}$ eV m

mass of an electron: $m_e = 9.11 \times 10^{-31}$ kg$= 0.511$ MeV$/c^2$

mass of a proton: $m_p = 1.67 \times 10^{-27}$ kg$= 938$ MeV$/c^2$

1 mile = 1.609 kilometers

1. A policeman tickets a driver for speeding, using a radar device to determine that the speed of the car was 70.0 mph$\pm$2.5 mph. Given that the mass of the car is 8.1×10^{2} kg, calculate the minimum uncertainty in the position of the car along its direction of travel. *Help: [S-5]*

2. In the ground state of a hydrogen atom, the electron has a characteristic probability density which describes the probability of locating the electron within a given radial region from the nucleus. If the FWHH of the probability density is approximately 0.05 nm, what is the minimum uncertainty in the radial component of the electron's momentum? *Help: [S-6]*

3. An early theory of the atomic nucleus held that protons and electrons existed in the nucleus in sufficient numbers to account for the total mass and charge of the nucleus. Since the nucleus is roughly 10^{-15} m in diameter, the positions of the protons and electrons would be known to within at least 10^{-15} m for any of the three Cartesian coordinates of position.

   a. Calculate the uncertainty in any Cartesian component of the momentum of an electron and of a proton in the nucleus. *Help: [S-7]*

   b. Assuming that the momentum components are greater than or roughly comparable to their uncertainties, i.e. $p_x \geq \Delta p_x$, etc., calculate the minimum kinetic energy of a proton and of an electron confined to the nucleus. *Help: [S-4]*

4. Calculate the percent uncertainty, $\Delta\nu/\nu$, in the frequency of the emitted photon when a hydrogen atom makes a transition from the fourth

excited state to the ground state, assuming that the lifetime of the fourth excited state is $\tau = 2.75 \times 10^{-9}$ s. *Help: [S-3]*

5. The "linewidth" of a laser pulse is defined as the uncertainty $\Delta\lambda$ in the wavelength of the laser "spectral line."

   a. Relate the linewidth of the spectral line to the "bandwidth," $\Delta\nu$, the uncertainty in the frequency of the line. *Help: [S-1]*

   b. If the time duration of the laser pulse is 1.00 nanosecond (10^{-9} s), use the energy-time uncertainty relation to estimate the linewidth of a 603 nm laser line.

Brief Answers:

1. $\Delta x \geq 5.8 \times 10^{-38}$ m *Help: [S-5]*
2. $\Delta p_r \geq 2.1 \times 10^{-24}$ kg m/s *Help: [S-6]*
3. a. for both: $\Delta p_x \approx \Delta p_y \approx \Delta p_z \geq 2.0 \times 10^8$ eV/c *Help: [S-7]*

   b. For the electron: $E_k \geq 117 \times 10^9$ eV $= 117$ GeV *Help: [S-8]*

   For the proton: $E_k \geq 6.4 \times 10^7$ eV $= 64$ MeV *Help: [S-9]*
4. $\Delta\nu/\nu = 1.84 \times 10^{-8} = 1.84 \times 10^{-6}$% *Help: [S-10]*
5. a. $\Delta\nu \approx (c\Delta\lambda)/\lambda^2$ *Help: [S-1]*

   b. $\Delta\lambda \approx 1.93 \times 10^{-13}$ m $= 0.193$ pm

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS, Problem 5)*

$$\begin{aligned}
\nu &= \frac{c}{\lambda} \\
\nu + \Delta\nu &= \frac{c}{\lambda + \Delta\lambda} \\
\Delta\nu &= \frac{c}{\lambda + \Delta\lambda} - \frac{c}{\lambda} = -\frac{c\Delta\lambda}{(\lambda + \Delta\lambda)\lambda} \simeq -\frac{c\Delta\lambda}{\lambda^2}
\end{aligned}$$

Hence, for a small spread of frequencies, that spread is:

$$|\Delta\nu| = c\Delta\lambda/\lambda^2.$$

This can also be obtained by differentiation.

S-2 *(from TX-2c)*

A function's root-mean-square deviation from its mean value is called its RMS deviation and is widely used in modern technology. For a function f(x) it is defined by:

$$\Delta f(x) = \sqrt{\overline{\left[f(x) - \overline{f(x)}\right]^2}}$$

where the overhead bar (an "overline") indicates the "mean" (average) value of the overlined quantity. To simplify the above expression we expand the square and take its average, noting that the average of $\overline{f(x)}$ is just itself (since it is already a number). Then:

$$\Delta f(x) = \sqrt{\overline{f^2(x)} - \overline{f(x)}^2}$$

This quantity is a measure of the width of the distribution function that describes the frequency of occurrence of a particular value of f(x). Thus a quantity $f(x)$ with a distribution function having a broad peak will have a larger RMS value than one with a narrower peak.

S-3 *(from PS-Problem 4)*

The lifetime-energy level width relationship states that

$$\Delta E\,\tau \simeq \hbar \Longrightarrow \Delta E \simeq \frac{\hbar}{\tau}.$$

The uncertainty in energy only applies to excited states, because the ground state has an infinite lifetime. The energy of the photon produced when an atomic system makes a transition from an excited energy state to the ground state state is:

$$E_{photon} = h\,\nu = E_n - E_1,$$

so the frequency ν will be uncertain by an amount $\Delta E/h$, where ΔE is the level width of the excited state. For one-electron atomic systems, $E_n = -E_0/n^2$, so for a transition from the fourth excited state to the ground state:

$$h\nu = E_5 - E_1 = -E_0/25 + E_0 = (24/25)\,E_0,$$

where $R_0 = 13.6\,\text{eV}$ for hydrogen, so:

$$\frac{\Delta\nu}{\nu} = \frac{(\Delta E/h)}{\nu} = \left(\frac{\Delta E}{h\nu}\right).$$

S-4 *(from PS, Problem 3b)*

The kinetic energy is given by $E_k = p^2/2m$ where m is the mass of the particle and $p^2 = p_x^2 + p_y^2 + p_z^2$.

S-5 *(from PS, Problem 1)*

$$\begin{aligned}\Delta x &\geq \frac{(6.63\times10^{-34}\,\text{J s})(3600\,\text{s/hr})(\,\text{kg m}^2\,\text{s}^{-2}/\,\text{J})}{(2)(2\pi)(8.1\times10^2\,\text{kg})(2.5\,\text{mi/hr})(1.609\times10^3\,\text{m/mi})}\\ &= 5.8\times10^{-38}\,\text{m}\end{aligned}$$

S-6 *(from PS, Problem 2)*

$$\Delta p_r \geq \frac{(6.63\times10^{-34}\,\text{J s})(\,\text{kg m}^2\,\text{s}^{-2}/\text{J})}{(2)(2\pi)(0.025\times10^{-9}\,\text{m})} = 2.1\times10^{-24}\,\text{kg m/s}$$

S-7 *(from PS, Problem 3a)*

$$\Delta p_x \geq \frac{197.3\times10^{-9}\,\text{eV m}/c}{(2)(1/2)(10^{-15}\,\text{m})} = 2.0\times10^{8}\,\text{eV}/c$$

S-8 *(from PS, Problem 3b)*

$$\Delta E_k \geq \frac{(3)(2.0\times10^8\,\text{eV}/c)^2}{(2)(0.511\times10^6\,\text{eV}/c^2)} = 117\times10^{9}\,\text{eV}$$

S-9 *(from PS, Problem 3b)*

$$\Delta E_k \geq \frac{(3)(2.0\times10^8\,\text{eV}/c)^2}{(2)(0.938\times10^9\,\text{eV}/c^2)} = 6.4\times10^{7}\,\text{eV}$$

S-10 *(from PS, Problem 4)*

from [S-3]:

$$\frac{4.14\times10^{-15}\,\text{eV s}}{(2\pi)(24/25)(13.6\,\text{eV})(2.75\times10^{-9})} = 1.84\times10^{-8}$$

MODEL EXAM

$\hbar c = 197.3 \times 10^{-9}\,\text{eV m}$

$c = 3.0 \times 10^{8}\,\text{m/s}$

1. See Output Skills K1-K3, this module's *ID Sheet*. The exam may include one or more of these skills, or none.

2. Suppose a state trooper measures the speed of a car and finds it to be (70.0 ± 2.5) mph which is (31.3 ± 1.1) m/s. Find the minimum uncertainty with which the trooper could simultaneously know the position of the car. Assume the car's weight to be 1790 lb, which means it has a mass of 8.1×10^{2} kg or $4.55 \times 10^{38}\,\text{eV}/c^2$.

3. The lifetime of the fourth excited state of the hydrogen atom is 2.75×10^{-9} s. Estimate the percentage spread in the frequencies of photons emitted when such atoms make a spontaneous transition from the first excited state to the ground state. The ground state energy of a hydrogen atom is $E_0 = -13.6\,\text{eV}$.

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, Problem 1.

3. See this module's *Problem Supplement*, Problem 4.