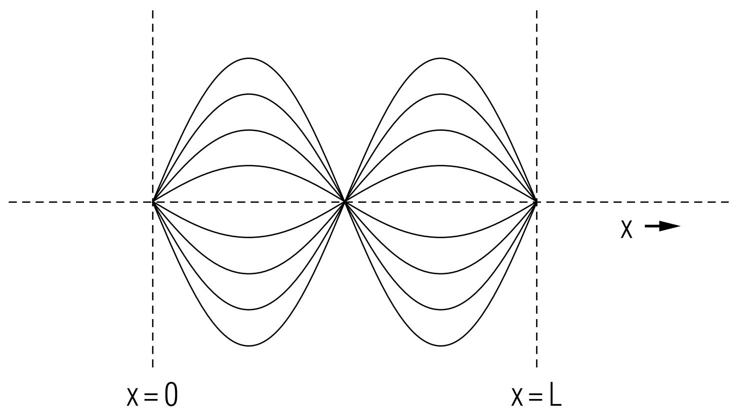
MISN-0-232

STANDING WAVES



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

STANDING WAVES

by

J. S. Kovacs, Michigan State University

Title: **Standing Waves**

Author: J. S. Kovacs, Michigan State University

Version: 5/7/2002 Evaluation: Stage 0

Length: 1 hr; 28 pages

Input Skills:

1. Vocabulary: amplitude, frequency, wavelength, wave equation, traveling wave (MISN-0-201); transverse wave, longitudinal wave, sound wave (0-202).
2. Write the wave equation for a one-dimensional traveling wave of given amplitude, frequency, wavelength, and direction of propagation (0-201).
3. Recall and use equations relating wave velocity, frequency, wavelength, and the characteristics of media: transverse waves along a string of given mass, length and tension; longitudinal waves in air and along a solid of given elastic constants (0-202).

Output Skills (Knowledge):

K1. Vocabulary: antinode, characteristic (resonant) frequency, first overtone, fundamental frequency, harmonics, node, normal modes, overtones, standing wave, superposition.

Output Skills (Problem Solving):

S1. Given a set of physical boundary conditions, determine the possible frequencies of the normal modes for transverse standing waves on a stretched string. For a string of given mass per unit length under a given tension, determine the numerical values for some of these frequencies.

S2. For longitudinal sound waves in a closed organ pipe, apply the appropriate boundary conditions to determine the possible frequencies for standing waves. Do the same for an open pipe. Determine the numerical values of the fundamental frequency and some of its lower harmonics.

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

PROJECT STAFF

Andrew Schnepp	Webmaster
Eugene Kales	Graphics
Peter Signell	Project Director

ADVISORY COMMITTEE

D. Alan Bromley	Yale University
E. Leonard Jossem	The Ohio State University
A. A. Strassenburg	S. U. N. Y., Stony Brook

Views expressed in a module are those of the module author(s) and are not necessarily those of other project participants.

© 2002, Peter Signell for Project PHYSNET, Physics-Astronomy Bldg., Mich. State Univ., E. Lansing, MI 48824; (517) 355-3784. For our liberal use policies see:

http://www.physnet.org/home/modules/license.html.

STANDING WAVES

by

J. S. Kovacs, Michigan State University

1. Introduction

1a. Properties of Running Waves. The velocity of a wave propagating through an elastic medium depends only upon properties of the medium itself,[1] and not upon properties of the externally induced vibrations which produce the wave. As an example, if you shake one end of a very long horizontally stretched string up and down, you can set up waves which travel down the length of the string with the stretched string's single characteristic velocity but with any wavelength you wish. Any wavelength can be made to travel along this "open" string (of infinite length) depending upon the driving frequency. Now if the string is of finite length, the waves, upon reaching the other end, undergo a reflection and travel back toward the source of the waves at the other end of the string. The superposition of these to-and-fro waves gives rise to destructive and constructive interference.

1b. Standing Waves and Normal Modes. For certain frequencies of excitation of a particular string, traveling waves will cease as standing waves set in. The frequencies at which this happens are called the strings "normal modes." For frequencies other than the normal mode frequencies, the propagating wave and its reflections produce an irregular motion of the string. Only the individual normal modes, or combinations of them, can be transversely excited and maintained on a string of finite length. This is the basis for the tuning of stringed instruments. Similar standing wave phenomena account for the excitation of only the characteristic frequencies of longitudinal sound waves in musical wind instruments, such as organ pipes.

1c. Effect of Boundary Conditions. The discrete (as opposed to continuous) nature of the possible frequencies for sustaining standing

[1] This is true at least for idealized systems which satisfy the classical wave equation. For example, the differential equation that describes transverse motion of a perfectly flexibly string is:

$$\partial^2\xi/\partial t^2 = (T/\mu)\,(\partial^2\xi/\partial x^2)\,.$$

Here T is the tension under which the string is held, μ is its mass per unit length, and the ratio T/μ is the square of the wave velocity. See "The Wave Equation and Its Solutions" (MISN-0-201).

waves on a string (or in a pipe) is attributed to the conditions that must be satisfied at the boundaries of the string (or pipe). The possible energy levels of atomic systems also have a discrete spectrum, instead of a continuous spectrum. The boundary conditions that must be satisfied by the solutions to such atomic problems can be seen to be the explanation of this discrete (quantized) spectrum.[2]

2. Standing Waves on a String

2a. Oppositely-Directed Waves Add Linearly. The solutions to the one-dimensional wave equation describing transverse waves on a stretched string allow for traveling waves moving in both directions along the string. A function $f(x-vt)$ describes a wave form propagating in the positive x-direction with velocity v, and $f(x+vt)$ describes one propagating in the negative x-direction.[3] One, or any linear combination of them, is a solution of:

$$\frac{\partial^2\xi}{\partial t^2} = v^2\frac{\partial^2\xi}{\partial x^2}\,, \tag{1}$$

independent of what the functional form "f" is. As the simplest case, the form could be that of a sine (or a cosine) function. Such a sine wave represents the propagation of a single frequency (and hence single wavelength) wave along the string. The expression

$$\xi(x,t) = \xi_0 \sin k(x-vt) = \xi_0 \sin(kx-\omega t)\,, \tag{2}$$

represents a wave of transverse displacement from equilibrium traveling to the right.[4] It satisfies Eq. (1) above and is a function of $x-vt$. If, at the same time, there were another wave traveling to the left, the resultant displacement from equilibrium at coordinate x at time t would be the superposition or the sum of the displacements due to the individual waves. Let's take the case where the two waves have amplitudes ξ_0 and ξ_0', and the same frequency. The superposition is

$$\xi(x,t) = \xi_0 \sin(kx-\omega t) + \xi_0' \sin(kx+\omega t)\,, \tag{3}$$

[2] This is developed and applied in MISN-0-242 and MISN-0-245.

[3] The shape of the wave form is determined by the function form f of the composite variables $x-vt$ or $x+vt$. See "The Wave Equation and Its Solutions" (MISN-0-201).

[4] Here ξ_0 is the amplitude of the wave (the maximum transverse displacement from equilibrium), k is 2π times the reciprocal of the wavelength, and ω is 2π times the frequency. These are the same as defined in MISN-0-201.

which can be written as[5]

$$\xi(x,t) = (\xi_0 + \xi_0')\sin kx\cos\omega t + (\xi_0' - \xi_0)\cos kx\sin\omega t\,. \qquad (4)$$

2b. Reflection Produces Oppositely Directed Waves. How can we generate waves which have the functional form of Eq. (4)? If a very long string is oscillated at one end (the other end is assumed to be an infinite distance away from the point of excitation), waves travel in one direction only, away from the oscillating end. If, however, the string is of finite length, the wave upon arriving at the far end will, in general, undergo a reflection, producing a traveling wave directed opposite to the incident one. Under steady state conditions, the two waves travel along the length of the string simultaneously, producing the wave form described by Eq. (4).

2c. Boundaries Determine the Possible Wavelengths. The superposition of the sinusoidal waves moving in the two directions along the string, as represented by Eq. (3) or Eq. (4), appears to have three free variables: the amplitudes ξ_0 and ξ_0' and the angular frequency ω.[6] You should be able to externally control the values of these parameters. The value of ω is determined by the frequency of transverse oscillation imposed at the input end of the string.[7] The amplitude ξ_0 is also determined by this input signal. The other amplitude ξ_0', however, is determined by the condition that for all times t the displacement of the string from its equilibrium position at the input end of the string (take this to be at $x = 0$) should be zero. According to Eq. (4),

$$\xi(0,t) = (\xi_0' - \xi_0)\,\sin\omega t\,, \qquad (5)$$

and this will be zero for all t only if $\xi_0' = \xi_0$. With that boundary condition, Eq. (4) is modified to:

$$\xi(x,t) = 2\xi_0\,\sin kx\,\cos\omega t\,. \qquad \text{(one end fixed)} \qquad (6)$$

The effect of this boundary condition is not a trivial one. Eq. (6) doesn't describe a traveling wave. Each point x on the string oscillates up and down with frequency ν (equal to $\omega/2\pi$). The amplitude of oscillation varies from point to point, the amplitude at x being $2\xi_0\sin kx$. Note also

[5]Using the trigonometric relations: $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$.

[6]The quantity k is related to ω through the fixed velocity of the waves along the string. Because $k = 2\pi/\lambda$, $\omega = 2\pi\nu$, and $\lambda\nu = v$, we have that $\omega = vk$.

[7]ξ_0 is essentially determined by the energy introduced at the input end. See MISN-0-203 for the relationship between the energy in a wave and its amplitude.

that all points oscillate in phase: the displacement at all points goes to zero at the same time (when ωt is an odd multiple of $\pi/2$) and has its maximum value at all points at the same time (when ωt is an integer multiple of π). These are standing waves as contrasted with the traveling waves which travel down the length of the "open" string when one end is excited.

2d. Second Boundary Condition: Restricted Frequencies. There is another boundary, the other end of the string, which also imposes a condition on the wave form, Eq. (6). If, for example, the other end at coordinate $x = L$ is also held fixed at the equilibrium position, this adds the restriction that

$$\xi(L,t) = 0, \qquad \text{for all } t\,. \qquad (7)$$

Setting ξ_0 equal to zero would satisfy this condition but it would result in the trivial uninteresting solution that $\xi(x,t)$ is zero for all x at all times t. The only other variable parameter is k. The quantity $\sin kx$, evaluated at $x = L$, will be zero only for certain values of k. Those are given by:

$$\sin kL = 0 \qquad \text{when} \qquad kL = \text{ integer multiple of } \pi\,, \qquad (8)$$

or, with $k = 2\pi/\lambda$,

$$2L = n\lambda, \qquad n = 1, 2, \ldots\,. \qquad (9)$$

This restricts the possible wavelengths of standing waves on the string (only those wavelengths for which an integer multiple is twice the string's length). This thus restricts the possible frequencies of excitation with which standing waves can be set up in a string which is fixed at the equilibrium position at both ends.

$$\nu_n = \frac{nv}{2L}, \qquad n = 1, 2, \ldots\,. \qquad \text{(both ends fixed)} \qquad (10)$$

where v is the wave velocity for waves along the string. *Help: [S-1]*

For example, the lowest frequency of standing wave that can be set up on a 25 cm string of mass 10 gm clamped at both ends and held under a tension of 100 N is 100 Hz. Other possible frequencies are integer multiples of this fundamental frequency, namely 200 Hz, 300 Hz, 400 Hz, ..., etc. Note that the fundamental mode corresponds to the case where exactly one-half wavelength fits between the fixed ends of the string. Other modes have a full wavelength, one and a half wavelengths, two wavelengths, etc.

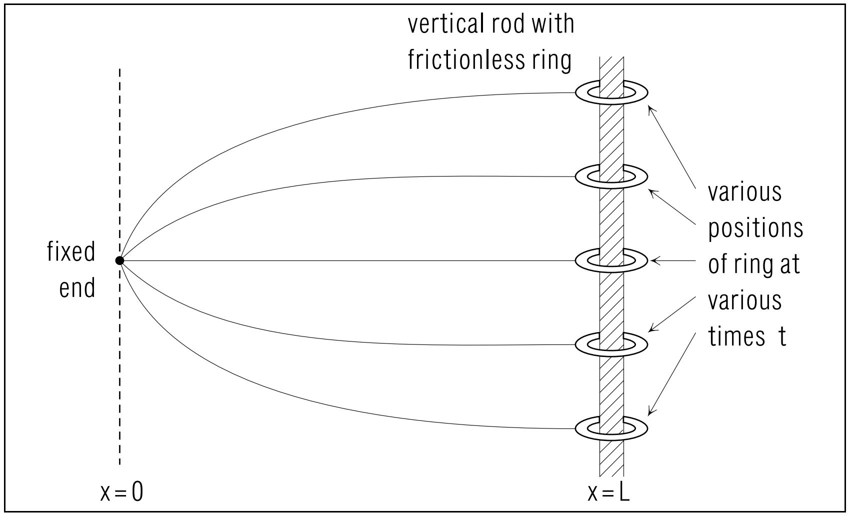


Figure 1. A stretched string connects a fixed point and a ring that can slide frictionlessly on a transverse rod.

2e. Other Conditions, Other Characteristic Frequencies. The characteristic frequencies that can excite standing waves along a string are determined by the physical boundary conditions. The set of frequencies as expressed by Eq. (10) above are characteristic of a string with both ends fixed. Another set of boundary conditions gives rise to a different set of characteristic frequencies. For example, if at the end $x = L$ the string is attached to a ring on a vertical frictionless rod (instead of being fixed), then this end is free to move up and down (see Fig. 1).

Furthermore, because the rod is frictionless, it cannot exert a vertical force on the ring and hence the ring cannot exert a vertical force on the string at the point of connection. Consequently, the tension in the string at $x = L$ must at all times be directed horizontally. A segment of flexible string, however, aligns itself along the direction of the net force on it. The alignment of the string at $x = L$ thus is at all times horizontal, its slope is always zero. This condition combined with the condition that the $x = 0$ end is fixed, yields the result that the characteristic frequencies are:

$$\nu_n = \frac{(2n-1)v}{4L}; \qquad n = 1, 2, \ldots . \qquad \textit{Help: [S-2]} \tag{11}$$

For example, for the string described after Eq. (10), if the end $x = L$ is attached to such a frictionless ring, the fundamental frequency is 50 Hz,

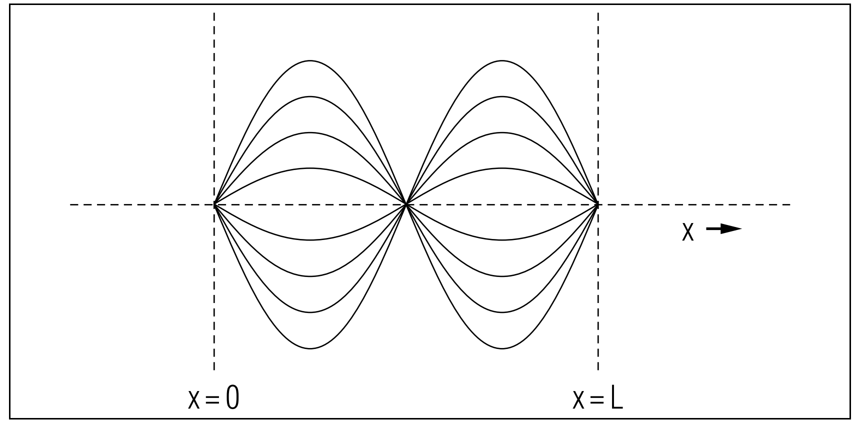


Figure 2. First overtone for standing waves on a string fixed at both ends.

corresponding to the case where the length of the system is a quarter wavelength of the wave as illustrated in Fig. 1.

2f. Nodes for Characteristic Frequencies. The normal modes of excitation for a string, of length L, fixed at both ends have the characteristic frequencies given by Eq. (10). For the second frequency of excitation, usually called the first overtone,[8] the length of the string L is exactly equal to the wavelength associated with that frequency. According to Eq. (6), the displacement from equilibrium at points x for any t is for the $n = 2$ mode given by:

$$\xi(x,t) = 2\xi_0 \sin\left[\frac{2\pi}{L}x\right] \cos\omega t\,. \tag{12}$$

This satisfies the boundary condition that for all times t the points $x = 0$ and $x = L$ have zero displacement from equilibrium. These non-moving points are called "nodal points" or "nodes." However, for this mode of excitation, the point $x = L/2$, the midpoint of the string, is also a node (see Fig. 2). If this point were also held fixed (along with $x = 0$ and $x = L$), this mode of standing wave could still be excited. The mode $n = 1$ could not be, nor could any mode which did not produce a node at $x = L/2$. Those modes which could be excited if the midpoint were

[8]The fundamental frequency is also called the first harmonic because it is the first member of a harmonic series. The second frequency is called the second harmonic or first overtone, etc, *Help: [S-3]*.

clamped are those with characteristic frequencies,

$$\nu_n = \frac{nv}{2L}; \qquad n = 2, 4, 6, \ldots . \tag{13}$$

3. Longitudinal Standing Waves

3a. Comparison to Transverse Waves. The differential equation satisfied by longitudinal displacement waves in one dimension is the same as for transverse waves, Eq. (1). The difference is in physical interpretation.[9] The longitudinal-wave solutions to Eq. (1), therefore, have a similar mathematical structure and, if the physical conditions are right, standing waves as represented by Eq. (6) can be established in such systems. This phenomenon is impressively illustrated in wind instruments such as organ pipes. Sound waves excited at one end of such pipes propagate to the other end and upon reflection interfere with the initial wave. Again, standing waves can be established for only certain characteristic frequencies of waves. These characteristic frequencies are determined by the length of the pipe and whether the boundary conditions force a node or an antinode[10] at each of the two ends.

3b. Open-End and Closed-End Pipes. The physical conditions that determine whether there is a node or an antinode at the end of a pipe are similar to the corresponding conditions for a node or antinode at the ends of a string (Sect. 2). A node will occur when the end of the pipe is closed, so that the displacement from equilibrium of an element of the medium—the air—is constrained to be zero at all times. On the basis of arguments, similar to those for which Fig. 1 is an illustration in the case of transverse waves on a string, an antinode will occur at the end of a pipe if it is open.[11] Figure 3 illustrates the amplitude distribution for one of the normal modes for standing waves in each case: (a) a pipe closed at one end and open at the other; and (b) a pipe open at both ends. *Help: [S-4]* In each case it is not the fundamental mode but the first overtone that is illustrated. Note that in case (a) the overtones, the tones that actually occur, are not consecutive members of the harmonic series. For this system, the first, second, third. etc., overtures are the third, fifth, seventh, etc., harmonics. The system illustrated in case (b) exhibits all harmonics.

[9]For longitudinal compressional waves, such as sound waves, the displacement from equilibrium $\xi(x,t)$ of an element of the medium is in the same direction that the wave propagates. The velocity of the wave, of course, depends upon different elastic and inertial properties of the medium than in the case of waves on a string. See "Sound Waves and Transverse Waves on a String" (MISN-0-202).

[10]A node is a point with zero displacement from equilibrium at all times. An antinode is a point of maximum displacement—but varying sinusoidally with time. Figure 1 illustrates the boundary condition that produces an antinode at one end of a string with standing waves.

[11]For a more detailed argument refer to Section 5.3, pp. 240-241 of *Waves*, the Berkeley Physics Course, F. S. Crawford, Jr., McGraw-Hill (1965). Also, the antinode at the end of an open pipe really occurs (1/3) of the pipe's diameter outside the open end. Thus the effective length of a pipe with an open end is (1/3) diameter longer than the physical length. For the purposes of this module we will assume the effective length is the physical length or that the diameter of the pipe is very much smaller than any wavelength of interest.

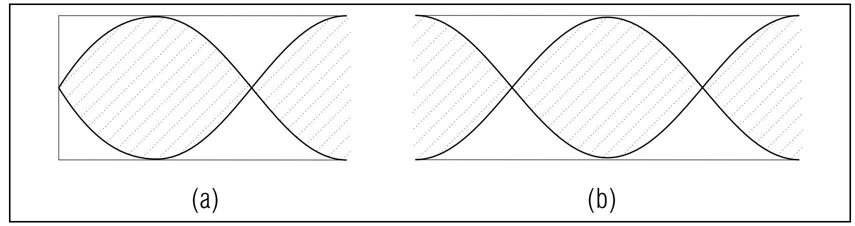


Figure 3. The amplitude distribution for standing sound waves in: (a) a pipe closed at one end and open at the other; and (b) a pipe open at both ends.

Acknowledgments

The author wishes to thank Professor James Linneman for several useful suggestons. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **anti-node**: a position along the medium where a standing wave shows maximum oscillation as time progresses.
- **boundary condition**: for a wave function, a requirement that it have some particular value at some point in space for all times. The requirement represents on a physical restraint that has been placed on the medium.

- **characteristic frequency**: for a system, any of the normal-mode frequencies.
- **fundamental frequency**: for a system, the lowest normal-mode frequency.
- **fundamental mode**: for a system, the normal mode having the lowest frequency.
- **harmonic series**: for a system, the set of frequencies that are integer multiples of the system's fundamental frequency. The lowest-frequency harmonic, the fundamental-mode frequency, is called the "first harmonic," the next highest is called the "second harmonic," etc. Not all of the system's harmonic-series frequencies need occur in the system's actual normal-mode frequencies.
- **longitudinal wave**: a wave whose oscillation is in the direction of the wave's direction of motion. Sound waves are longitudinal waves.
- **node**: a position along the medium where a standing wave shows zero oscillation at all times.
- **normal mode**: for a wave function, a physical situation in which a standing wave occurs with a single frequency of oscillation. For one system, different frequencies can occur under different stimuli of the medium. Such different oscillations are called different normal modes of the system. The collection of all such possible modes are called t"the system's normal modes."
- **overtone**: for a system, any normal-mode frequency other than the lowest one. The lowest-frequency overtone is called the "first overtone," the next highest the "second overtone," etc.
- **running wave**: a wave for which crests (for example) travel past any given location as they move along the medium. Thus successive photographs of the medium show a wave moving along it.
- **standing wave**: a wave which does not move along the medium. Successive photographs of the medium show a stationary wave oscillating "in place." This means that at some positions, called "nodes", the standing wave never varies from zero as time progresses. At all other positions it varies smoothly from a crest to zero to a trough to zero to a crest, etc, as time progresses. At positions half-way between nodes, the crests and troughs are a maximum and these are called "anti-nodes."

- **transverse wave**: a wave whose oscillation is at right angles to the wave's direction of motion. String waves and light waves are transverse waves.
- **travelling wave**: a wave for which crests (for example) travel past any given location as they move along the medium. Thus successive photographs of the medium show a wave moving along it.

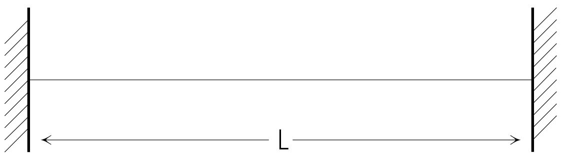
PROBLEM SUPPLEMENT

$$v = (T/\mu)^{1/2}$$

Note 1: Work these problems in order, completing each one successfully before going on to the next.

Note 2: Problems 5 and 8 also occur in this module's *Model Exam*.

1.



A stretched steel wire, 75.0 cm long, is fixed at both ends to perfectly reflecting walls. The wire is under a tension of 3.00×10^2 N and has a mass of 0.625 gm.

a. Calculate the velocity of propagation of any transverse wave existing on this wire. *Help: [S-9]*

b. Taking one fixed end as $x = 0$ and the other as $x = L$, apply the boundary conditions of requiring a node at both ends to determine the wavelengths of the fundamental and the first and second overtones. Sketch the amplitude distributions of these three normal modes.

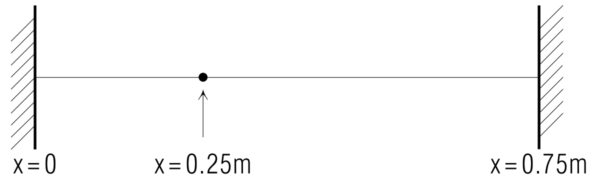
c. Compute the corresponding frequencies of these normal modes and determine whether or not they form a series of harmonics. If so label each accordingly. *Help: [S-10]*

2. Suppose the sketched wire in Problem 1 is clamped at its midpoint, thus requiring a node to exist there.

a. Apply these new boundary conditions to determine the wavelengths of the first three normal modes. Again, sketch the amplitude distributions. *Help: [S-5]*

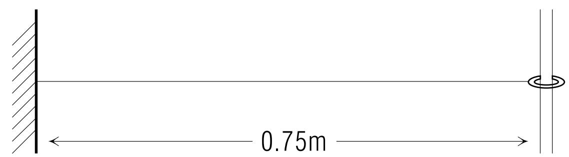
b. Determine the frequencies of these three normal modes and whether or not they form a series of harmonics. If so, label them as in Problem 1.

3.



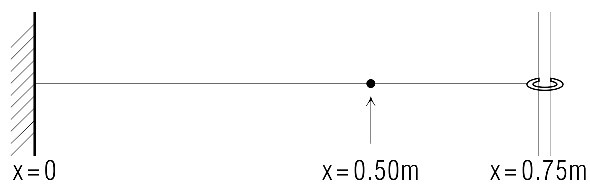
Now suppose the wire in Problem 1 is clamped 1/3 of the way from the end $x = 0$, as illustrated in the sketch. Repeat parts (a) and (b) of Problem 2 with this new set of boundary conditions, namely that any normal modes must possess nodes at $x = 0$, $x = 0.25$ m and $x = 0.75$ m. *Help: [S-5]*

4.



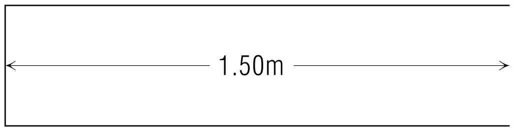
Replace one of the fixed ends of the steel wire in Problem 1 with a frictionless ring on a rod, so that a node always occurs at the left end and an antinode always occurs at the right end. Assuming all other conditions remain the same (tension in the wire, mass per unit length of the wire), repeat parts (a) and (b) of Problem 2 with this set of boundary conditions.

5.



Now suppose the wire in Problem 4 is clamped 0.25 m from the free end, as illustrated in the sketch. Repeat parts (a) and (b) of Problem 2 with this set of boundary conditions (nodes at $x = 0$ and $x = 0.50$ m and an antinode at $x = 0.75$ m). *Help: [S-5]*

6.



An organ pipe, 1.50 m long, is closed at one end and open at the other. Thus for whatever standing wave patterns that are excited in the pipe, a node must occur at the closed end and an antinode must occur at the open end.

a. Calculate the speed of sound waves in air at 20 °C. Use M = 28.8 gm/mole and $\gamma = 1.40$ for air, and $R = 8.31\,\mathrm{J/(mole\,K)}$ is the ideal gas constant. *Help: [S-8]*

b. Use the boundary conditions to determine the wavelengths of the fundamental and the first and second overtones. Sketch the air pressure amplitude distribution of these three normal modes.

c. Compute the corresponding frequencies of these three normal modes and determine whether or not they form a series of harmonics. If so label each accordingly.

7. Replace the organ pipe of Problem 6 with a pipe of equal length but open at both ends. This condition requires that an antinode occur at both ends of the pipe for any standing wave. Assuming all external conditions remain the same;

a. Apply the boundary conditions to determine the wavelengths of the first three normal modes and sketch the air pressure amplitude distributions. *Help: [S-6]*

b. Compute the corresponding frequencies and label them as harmonics.

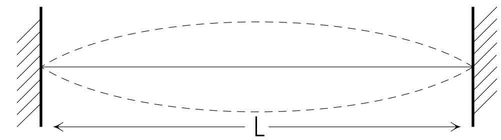
8. A hole is drilled in the pipe of Problem 7 halfway from either end, thus imposing antinode at the midpoint of the pipe for any allowed normal mode. Repeat parts (a) and (b) of Problem 7 for this new system.

Brief Answers:

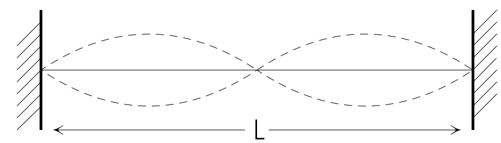
1. a. $v = 600\,\mathrm{m/s}$

b. $kL = n\pi\,, \qquad n = 1, 2, 3, \ldots$

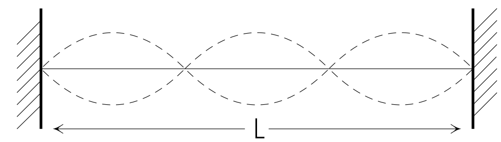
Fundamental:
$\lambda = 2\,L = 1.50\,\mathrm{m}$



First Overtone:
$\lambda = L = 0.75\,\mathrm{m}$



Second Overtone:
$\lambda = 2/3\,L = 0.50\,\mathrm{m}$



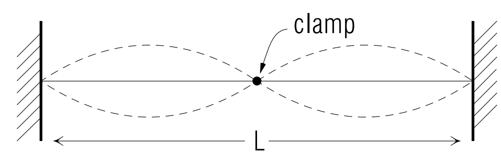
c. Fundamental: $\nu = 400\,\mathrm{Hz}$ (1st Harmonic)

First Overtone: $\nu = 800\,\mathrm{Hz}$ (2nd Harmonic)

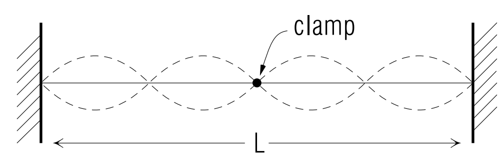
Second Overtone: $\nu = 1200\,\mathrm{Hz}$ (3rd Harmonic)

2. a. $k\,L/2 = n\pi\,, n = 1, 2, 3, \ldots$

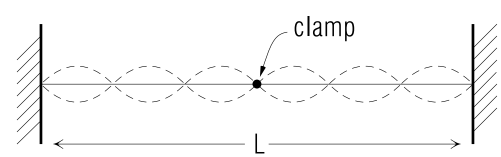
Fundamental:
$\lambda = L = 0.75\,\mathrm{m}$



First Overtone:
$\lambda = 1/2\,L = 0.375\,\mathrm{m}$



Second Overtone:
$\lambda = 1/3\,L = 0.25\,\mathrm{m}$



b. Fundamental: $\nu = 800\,\text{Hz}$ (1st Harmonic)

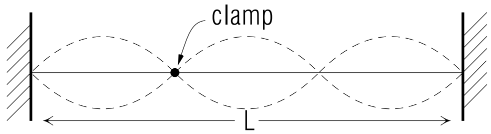
First Overtone: $\nu = 1600\,\text{Hz}$ (2nd Harmonic)

Second Overtone: $\nu = 2400\,\text{Hz}$ (3rd Harmonic)

3. a. $k\dfrac{L}{3} = n\pi, \qquad n = 1, 2, 3, \ldots$

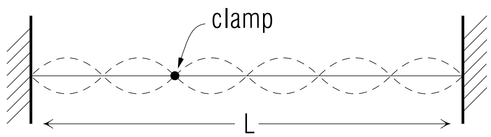
Fundamental:

$\lambda = 2/3\,L = 0.5\,\text{m}$



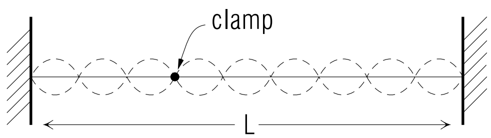
First Overtone:

$\lambda = 1/3\,L = 0.25\,\text{m}$



Second Overtone:

$\lambda = 2/9\,L = 0.166\,\text{m}$



b. Fundamental: $\nu = 1200\,\text{Hz}$ (lst Harmonic)

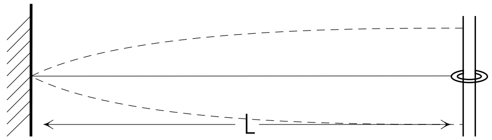
First Overtone: $\nu = 2400\,\text{Hz}$ (2nd Harmonic)

Second Overtone: $\nu = 3600\,\text{Hz}$ (3rd Harmonic)

4. a. $kL = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots, (\dfrac{2n-1}{2})\pi, \qquad$ for $n = 1, 2, 3, \ldots$

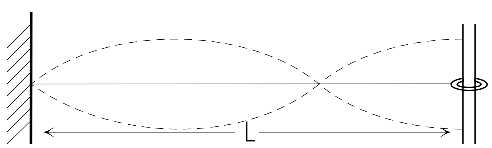
Fundamental:

$\lambda = 4\,L = 3.00\,\text{m}$



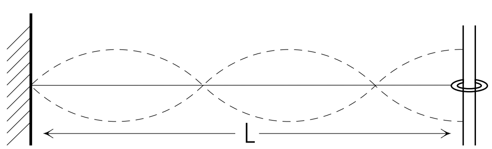
First Overtone:

$\lambda = 4/3\,L = 1.00\,\text{m}$



Second Overtone:

$\lambda = 4/5\,L = 0.60\,\text{m}$



b. Fundamental: $\nu = 200\,\text{Hz}$ (1st Harmonic)

First Overtone: $\nu = 600\,\text{Hz}$ (3rd Harmonic)

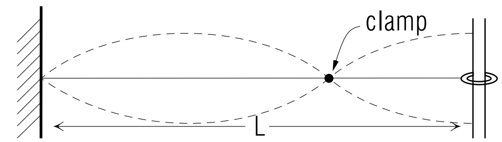
Second Overtone: $\nu = 1000\,\text{Hz}$ (5th Harmonic)

(Note that for this set of boundary conditions that even harmonics do not exist.)

5. a. $k\,2L/3 = n\pi\,, n = 1, 2, 3, \ldots$

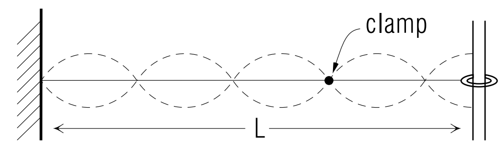
Fundamental:

$\lambda = 4/3\,L = 1.00\,\text{m}$



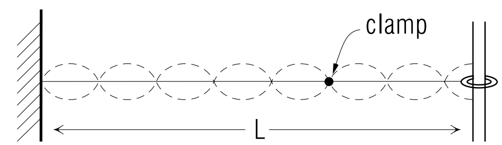
First Overtone:

$\lambda = 4/9\,L = 0.33\,\text{m}$



Second Overtone:

$\lambda = 4/15\,L = 0.20\,\text{m}$



b. Fundamental: $\nu = 600\,\text{Hz}$ (1st Harmonic)

First Overtone: $\nu = 1800\,\text{Hz}$ (3rd Harmonic)

Second Overtone: $\nu = 3000\,\text{Hz}$ (5th Harmonic)

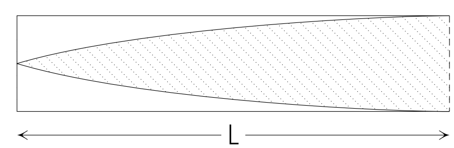
(Again note that for this set of boundary conditions only odd harmonics are present.)

6. a. $v = 344\,\text{m/s}$

b. $kL = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots (\dfrac{2n-1}{2})\pi, \qquad$ for $n = 1, 2, 3, \ldots$

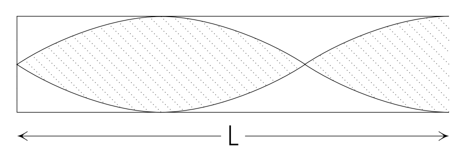
Fundamental:

$\lambda = 4\,L = 6.00\,\text{m}$

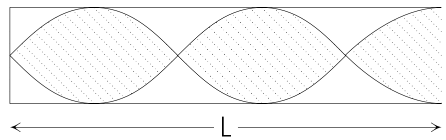


First Overtone:

$\lambda = 4/3\,L = 2.00\,\text{m}$



Second Overtone:
$\lambda = 4/5\,L = 1.20\,\mathrm{m}$



c. Fundamental: $\nu = 57.3\,\mathrm{Hz}$ (1st Harmonic)

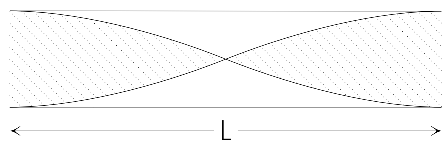
First Overtone: $\nu = 172\,\mathrm{Hz}$ (3rd Harmonic)

Second Overtone: $\nu = 286.7\,\mathrm{Hz}$ (5th Harmonic)

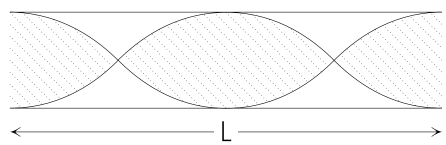
(Note that this system is entirely analogous to the stretched string in Problem 4; the boundary conditions are the same.)

7. a. $k\,L = n\pi\,, \qquad n = 1, 2, 3, \ldots.$ *Help: [S-6]*

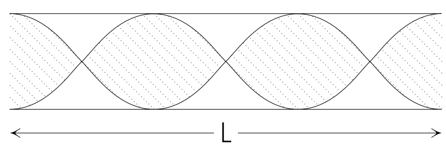
Fundamental:
$\lambda = 2\,L = \ 3.00\,\mathrm{m}$



First Overtone:
$\lambda = L = 1.50\,\mathrm{m}$



Second Overtone:
$\lambda = 2/3\,L = 1.00\,\mathrm{m}$



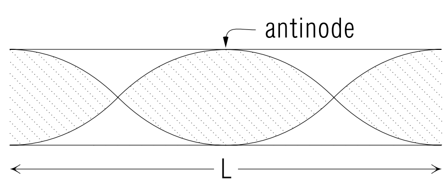
b. Fundamental: $\nu = 114.7\,\mathrm{Hz}$ (1st Harmonic)

First Overtone: $\nu = 229.3\,\mathrm{Hz}$ (2nd Harmonic)

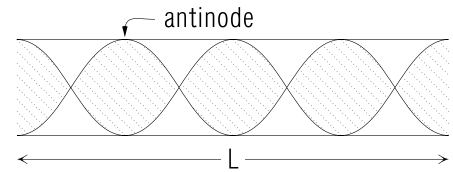
Second Overtone: $\nu = 344\,\mathrm{Hz}$ (3rd Harmonic)

8. a. $k(\frac{L}{2}) = n\pi, \qquad n = 1, 2, 3, \ldots$

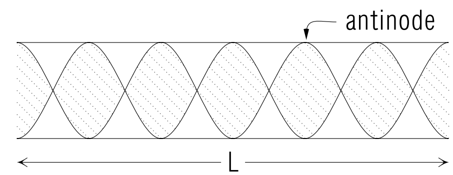
Fundamental:
$\lambda = L = 1.50\,\mathrm{m}$



First Overtone:
$\lambda = L/2 = 0.75\,\mathrm{m}$



Second Overtone:
$\lambda = L/3 = 0.50\,\mathrm{m}$



b. Fundamental: $\nu = 229.3\,\mathrm{Hz}$ (1st Harmonic):

First Overtone: $\nu = 458.7\,\mathrm{Hz}$ (2nd Harmonic)

Second Overtone: $\nu = 688\,\mathrm{Hz}$ (3rd Harmonic)

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-2d)*

Avoid memorizing specific formulae for normal mode frequencies since there are so many, essentially one for each set of boundary conditions. Rather, apply the boundary conditions for the particular problem at hand to Eq. (6) and derive a unique formula for the characteristic frequencies, just as was done for the case of a stretched string fixed at both ends, resulting in Eq. (10).

S-2 *(from TX-2e)*

At $x = 0$: $\sin kx = 0$, regardless of k.
At $x = L$, $\sin kL = 1$ since the string may experience its maximum displacement at this point: $\xi = (2\xi_0 \sin kL)\cos\omega t$.
The amplitude of the standing wave is $2\xi_0 \sin kL$. What values of kL (in radians) will yield a value of 1 for $\sin kL$?

S-3 *(from TX-2f)*

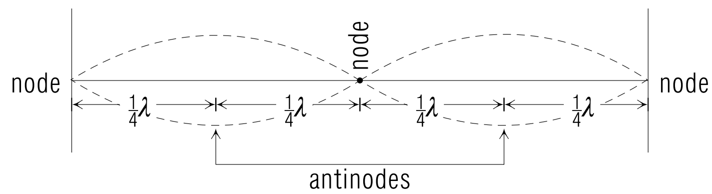
For a set of discrete characteristic frequencies corresponding to the normal modes of standing waves in any system, the lowest allowed frequency is always be called the "fundamental" frequency. The next highest allowed frequency is called the "first overtone," followed by the "second overtone," "third overtone," etc. If these overtones can be expressed as an integer, n, times the fundamental frequency, ν_0, i.e. $\nu_n = n\,\nu_0$, then the normal modes are said to constitute a series of "harmonics," where ν_n is the nth harmonic. The correspondence between overtones and harmonics is not fixed since in certain systems some harmonics are not allowed by the boundary conditions. *Help: [S-7]*

S-4 *(from TX-3b)*

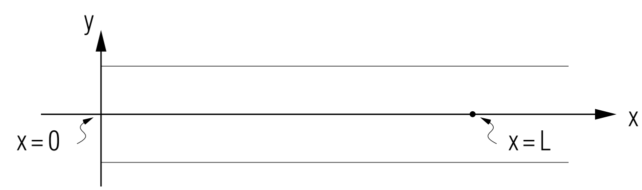
The illustration of the amplitude distribution for standing waves in pipes depicted in Fig. 3 is not intended to be a literal picture of what you would see in an organ pipe. Rather it relies on an analogy with the standing wave patterns set up in stretched strings.

S-5 *(from PS-problems 2, 3, 5)*

Remember that only normal modes which naturally have a node at the position of the clamp satisfy the boundary conditions, hence are the only ones that can exist. Furthermore there are regular spacings between nodes and antinodes: 1/2 wavelength between adjacent nodes, 1/2 wavelength between adjacent antinodes, and 1/4 wavelength between adjacent nodes and antinodes.



S-6 *(from PS-problem 7)*



Since antinodes occur at both ends of an open pipe (x = 0 and x = L), the original waves must be represented by functions which do not cancel automatically at x = 0. Since the waves are sinusoidal, cosine functions may be used as well as sine functions to fulfill the above condition:
$\xi = \xi_1 + \xi_2 = \xi_0\cos(kx - \omega t) + \xi_0\cos(kx + \omega t) = 2\xi_0 \cos kx \cos\omega t$.
This equation for the standing wave does not yield a node at $x = 0$. It satisfies the boundary conditions of the system at hand.

S-7 *(from [S-3])*

For example, the first overtone in the case of the string fixed at both ends is also the 2nd harmonic, but for a string fixed at one end and free at the other, the first overtone becomes the 3rd harmonic (work through the allowed frequencies established by Eq. (10) and Eq. (11) to convince yourself of this). Furthermore, some systems do not exhibit harmonics at all since the overtones cannot be expressed as integral multiples of the fundamental.

S-8 *(from PS-problem 6)*

The speed of sound is related to the given quantities in *Sound Waves and Small Transverse Waves on a String*, MISN-0-202, Sect. 5d.

S-9 *(from PS-problem 1a)*

If you don't know how to get the velocity from the given quantities, you failed to learn it in a prerequisite module (see the Input Skills in this module's *ID Sheet*). It is also possible to recall how to do it as you read the text of this module. As you work with the numbers, make sure you convert all quantities to the proper SI units (m, kg, s) and do units algebra using, if necessary, $N = kg\,m/s^2$.

S-10 *(from PS-problem 1c)*

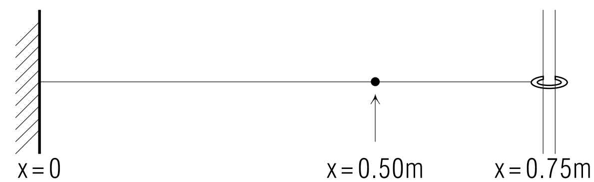
If you don't know how to get frequency from the other quantities, you failed to learn it in a prerequisite module (see the Input Skills in this module's *ID Sheet*).

MODEL EXAM

$$v = (T/\mu)^{1/2}$$

1. See Output Skill K1 in this module's *ID Sheet*.

2.



A stretched steel wire, 75.0 cm long, is fixed at one end to a perfectly reflecting wall. The other end is attached to a ring that can slide frictionlessly on a transverse rod, so this end is said to be "free." The wire is under a tension of 3.00×10^2 N and has a mass of 0.625 gm. The wire is clamped 0.25 m from the free end, as illustrated in the sketch.

a. Determine the wavelengths of the first three normal modes. Sketch the amplitude distributions.

b. Determine the frequencies of these three normal modes and determine whether or not they form a series of harmonics. If they do, label them properly.

3. An organ pipe, 1.50 m long, is open at both ends. The temperature of the air is 20 °C. A hole is drilled halfway between the ends.

a. Determine the wavelengths of the first three normal modes and sketch the air pressure amplitude distributions.

b. Compute the corresponding frequencies and label them as harmonics.

Brief Answers:

1. See this module's *text*.
2. See this module's *Problem Supplement*, problem 5.
3. See this module's *Problem Supplement*, problem 8.