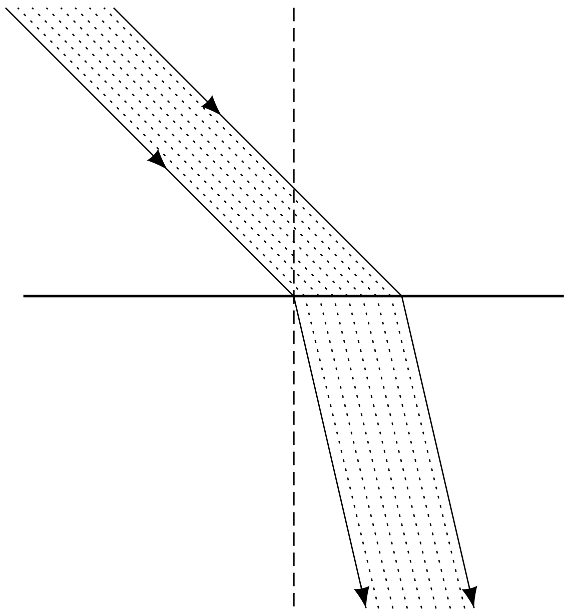
MISN-0-225

BREWSTER'S LAW AND POLARIZATION



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

BREWSTER'S LAW AND POLARIZATION

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Title: **Brewster's Law and Polarization**

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Input Skills:

1. Skills from "Derivation From Maxwell's Equations" (MISN-0-210).
2. Skills from "Energy and Momentum in an Electromagnetic Wave" (MISN-0-211).
3. Skills from "The Rules of Geometrical Optics" (MISN-0-220).

Output Skills (Knowledge):

K1. Use the results obtained from Maxwell's Equations, tabulated in Eq. 26.13 of AF (see External Resouces, Item 1, below) to:
  a. Explain the phenomenon of polarization by reflection.
  b. Show how the conditions for complete polarization arise (explain how Brewster's Law comes about).

Output Skills (Rule Application):

R1. Use Eqs. 26.13 of AF (see Item 1 of Required External Resouces, below) to calculate the intensity of reflected and transmitted radiation for some simple geometrical situations (such as normal incidence and at the polarizing angle).

External Resources (Required):

1. M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading (1970). For access, see this module's *Local Guide*.

External Resources (Optional):

1. J. A. Stratton, *Electromagnetic Theory*, Mc-Graw Hill, N.Y. (1941). For access, see this module's *Local Guide*.

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1. Description

While Snell's Law and the reflection law give you the directions of the refracted and the reflected rays when light is incident upon an interface between two optical media, you need to look at the electromagnetic nature of light to find out how much of the light is reflected and how much is refracted. The equations which give this can be derived from Maxwell's Equations and they give you the magnitudes of the electric field vector components in the reflected and refracted waves relative to the electric field components of the incident wave. This module deals with the application of these equations and, in particular, how polarization of light by reflection comes about.

2. Study Suggestions, Problems

- In AF,[1] study Section 26.6. You will need to understand Figures 26.14, 26.15, and 26.16 to know what's going on, so study these carefully.
- Answer study questions 9 and 10 (p. 639).
- Work through the worked-out example, 26.2, on p. 626, but note that $T_\pi = 0.766$, not 0.442, and $R_\pi = -0.165$).
- Work through the illustrative problem that follows in Sect. 4.
- Then work AF's problems 26.7, 26.8, 26.9, 26.10, 26.11, 26.12, and 26.13 (but see the discussion of problem 26.13 in Sect. 3 below before you work it).
- Work Problem A in Sect. 5 of this module.

[1] M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading (1970). For access, see this module's *Local Guide*.

3. Comments

We drop the inconsistent notation of AF and use the subscript "t" for the transmitted amplitudes and "r" for the reflected amplitudes.

- Eqs. 26.13 relate the amplitudes of the incident $\vec{E}$ and $\vec{B}$ field (with components $E_{i,\pi}$, $E_{i,\sigma}$, and $B_{i,\pi}$, $B_{i,\sigma}$ respectively) to the amplitudes of the reflected wave (with components $E_{r,\pi}$, $E_{r,\sigma}$ and $B_{r,\pi}$, $B_{r,\sigma}$), and to the amplitudes of the transmitted wave (with components $E_{t,\pi}$, $E_{t,\sigma}$ and $B_{t,\pi}$, $B_{t,\sigma}$).
- In problem 26.13 you are asked to work on energy flow. AF's derivation of energy flow in an electromagnetic wave can be summed up this way: you add the energy densities for static electric and magnetic fields and multiply the sum by the speed of the waves, c. That derivation is erroneous: one cannot transform from a static frame to an ultrarelativistic one that way.

  We write $\overline{S}$ for the total time-average energy intensity in an electromagnetic wave, the energy transported across unit cross-sectional area per unit time. Then the correct answer for $\overline{S}$ is:[2]

  $$\overline{S} = \frac{1}{2} n c \epsilon_0 E^2,$$

  where E is the amplitude of the electric field vector in the wave, n is the index of refraction of the medium through which the wave is passing, c is the speed of light in vacuum, and $\epsilon_0 \equiv 1/(4\pi k_e)$ where $k_e \equiv 10^{-7} c^2\,\mathrm{N\,m^2\,C^{-2}}$.

  In order to produce the fraction of the power P (energy per unit time) in the incident beam that is in the outgoing transmitted beam, we must take into account the fact that the energy in the transmitted beam covers a larger cross-sectional area than in the incident or reflected beams (to get energy per unit time you must multiply the intensity by the cross-sectional area).

  The cross-sectional area of the transmitted beam is larger than that of the incident beam by the factor:

  $$\frac{A_t}{A_i} = \frac{\cos\theta_r}{\cos\theta_i},$$

[2] J. A. Stratton, *Electromagnetic Theory*, Mc-Graw Hill, N.Y., 1941. For access, see this module's *Local Guide*.Note, however, that Stratton's $\overline{S}$ is only correct if you first multiply it by $(1/\mu_0)$.

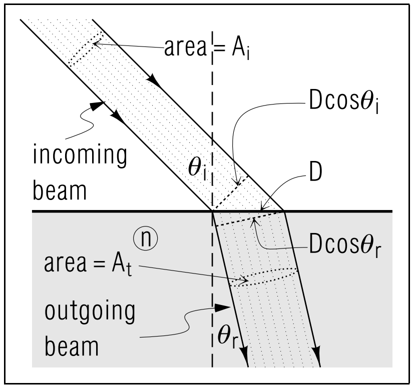

Figure 1. Refraction of a beam with incident cross-sectional area A_i and transmitted area A_t.

which, at the Brewster angle, is equal to n, the index of refraction of the refracting medium (this can be deduced from Fig. 1).

4. An Illustrative Problem

4a. The Problem. This solved exercise illustrates all of the Output Skills of this module.

Light of intensity $\overline{S_i}$ is incident at the polarizing angle at the surface of a piece of glass of index of refraction n. Determine the intensity of the reflected and refracted waves and show that energy is conserved.

4b. Solution. In the incident wave there are, in general, contributions from various directions of polarization perpendicular to the direction of propagation. Each of these polarization vectors (directions of $\vec{E}$) can be written as a linear superposition of two mutually perpendicular components along any two orthogonal directions perpendicular to the propagation direction. These two directions we will conveniently take as the π-direction and the σ-direction of AF's Figs. 26.14-16. The intensity of the incident wave will be just the sum of the intensities of the π- and σ-components).

Call the total incident intensity $\overline{S_i}$:

$$\overline{S_i} = c\epsilon_0 (E_{i,\pi}^2 + E_{i,\sigma}^2).$$

For convenience and interest we break this into incident π and σ intensities:

$$\overline{S_i} = \overline{S_{i,\pi}} + \overline{S_{i,\sigma}}.$$

where:

$$\begin{aligned} \overline{S_{i,\pi}} &\equiv c\epsilon_0 E_{i,\pi}^2 \\ \overline{S_{i,\sigma}} &\equiv c\epsilon_0 E_{i,\sigma}^2 \end{aligned}$$

Because the light is incident at the polarizing angle, $\theta_i + \theta_r = \pi/2$ (see shaded section, AF's p. 626, and $\tan\theta_i = n$ (these are the essential statements of Brewster's Law proved at the top of AF's p. 626), we have:

$$\tan(\theta_i + \theta_r) = \infty,$$

so:

$$R_\pi \equiv \frac{E_{r,\pi}}{E_{i,\pi}} = 0.$$

For the other reflection coefficient we use $\tan\theta_i = n$ and $\sin\theta_i = \dfrac{n}{\sqrt{n^2 - 1}}$ to get:

$$\begin{aligned} R_\sigma &\equiv \frac{E_{r,\sigma}}{E_{i,\sigma}} = \frac{\sin(\theta_r - \theta_i)}{\sin(\theta_r + \theta_i)} = \sin(\theta_r - \theta_i) \\ &= \sin\theta_r \cos\theta_i - \cos\theta_r \sin\theta_i = \cos^2\theta_i - \sin^2\theta_i = 1 - 2\sin^2\theta \\ &= \frac{1 - n^2}{1 + n^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} T_\pi &\equiv \frac{E_{t,\pi}}{E_{i,\pi}} = \frac{1}{n} \\ T_\sigma &\equiv \frac{E_{t,\sigma}}{E_{i,\sigma}} = \frac{2}{1 + n^2}. \end{aligned}$$

These are all for the special case of incidence at the polarizing angle. (For other angles, except $\theta_i = 0, \theta_r = 0$, these coefficients wouldn't be so simply expressible in terms of n.)

The ratio of the transmitted to incident π intensity is:

$$\frac{\overline{S_{t,\pi}}}{\overline{S_{i,\pi}}} = \frac{nc\epsilon_0 T_\pi^2}{2} = nT_\pi^2 = \frac{1}{n}.$$

Similarly,

$$\frac{\overline{S_{t,\sigma}}}{\overline{S_{i,\sigma}}} = \frac{nc\epsilon_0 T_\sigma^2}{2} = n\,\frac{4}{(1+n^2)^2},$$
$$\frac{\overline{S_{r,\pi}}}{\overline{S_{i,\pi}}} = 0,$$
$$\frac{\overline{S_{r,\sigma}}}{\overline{S_{i,\sigma}}} = \frac{(1-n^2)^2}{(1+n^2)^2}.$$

Finally, to get the ratios of outgoing to incoming energy flows, labeled P, we multiply the transmitted intensities by the area-ratio n and add the reflected intensities:

$$\frac{P_{f,\pi}}{P_{i,\pi}} = 1 + 0 = 1; \qquad \frac{P_{f,\sigma}}{P_{i,\sigma}} = n^2\frac{4}{(1+n^2)^2} + \frac{(1-n^2)^2}{(1+n^2)^2} = 1.$$

▷ For the case of light incident upon a plane interface from air to water ($n = 1.333$), with a 30.0° angle of incidence, show that: (1) the angle of refraction is 22.0°; (2) the percent of energy (for each state of polarization) in the reflected and transmitted waves 1.19% of the energy in the wave is reflected while 98.81% is transmitted (at an angle of refraction of 22°).

If the light is unpolarized, the reflected intensity is the *average* of $R(E_\parallel)$ and $R(E_\perp)$, etc. Note that between 30° and 60° incidence the reflected intensity is mostly of light polarized in the $E_\perp$ mode, so that if you wear polaroid glasses oriented so that they *absorb* $E_\perp$-light, you'll notice a considerable decrease in "glare."

5. Problem

A

Use the results of AF's Problem 26.8 to fill in the table for $\theta_i = 0$.

6. Problem Answers, Comments

26.7

a. $R_\pi = 0.092$, $T_\pi = 0.728$.

b. $R_\sigma - 0.303$, $T_\sigma = 0.697$.

26.8

For normal incidence $\sin\theta_i = 0$, $\sin\theta_r = 0$. However, do not just set θ_i equal to zero or you will get indeterminate relations. Instead, call θ small so you can use $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and $\tan\theta \approx \theta$. Also use Snell's Law. You will find that the θ's cancel out, so your answer will be good in the limit as $\theta \to 0$. Note that, when you sketch the fields, for $n_1 < n_2$ there is a "phase-change" at the interface.

26.9

a. O.K. in AF.

b. $R = 0.2$, $T = 1.2$ (in all cases the reflection coefficient is the same in magnitude whether you go from medium 1 to medium 2, or from medium 2 to medium 1).

c. O.K. in AF. The change of phase upon reflection is illustrated in Fig. 26.14 where the reflected $\vec{E}$ and $\vec{B}$ waves reverse discontinuously upon reflection when the medium from which the wave reflects has higher index of refraction than the medium in which the reflection takes place.

26.10

$\tan\theta_i = 1.5$, so $\sin\theta_i = \dfrac{3}{\sqrt{13}}$; $\sin\theta_r = \dfrac{2}{\sqrt{13}}$.

26.11

If the critical angle is 45°, $n = \sqrt{2}/2$. The polarizing angle (the angle of incidence for which the reflected wave is plane polarized) is the angle whose tangent is $\sqrt{2}/2$, $\theta_i = 35°$.

26.12

$\tan\theta_i = 4/3$ (or 1.33) determines the angle of incidence θ_i. The angle above the horizontal is $\pi/2 - \theta_i = 37°$. The plane of polarization of the $\vec{E}$-vector is parallel to the surface of the water. (So if your polaroid glasses absorb light of this plane of polarization, and transmit light with polarization perpendicular to this, your glasses will very effectively cut down "glare" from this surface because there is very little radiation reflected from the surface with the polarization vector in the direction that is transmitted by the glasses.)

26.13

See Sects. 3 and 4.

a. $\overline{S_{i,\pi}} = c\epsilon_0 E_0^2$.

b. $E_{t,\pi} = E_0/n$.

c. $\overline{S_{t,\pi}} = \frac{1}{n} c\epsilon_0 E_0^2$.

d. There is no reflected wave so all the power must go from the incident wave to the transmitted wave. Multiplying the transmitted intensity by the ratio of the beam areas to get the power, we find the outgoing power equal to the incoming power.

Using these results, Problem 26.8 and find the intensity of the transmitted and reflected waves when the incident wave has intensity 1 for glass of $n = 1.5$. [Answer: The reflected intensity is 4% of the incident intensity, the transmitted intensity, 96%.]

Acknowledgments

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LOCAL GUIDE

The readings for this unit are on reserve for you in the Physics-Astronomy Library, Room 230 in the Physics-Astronomy Building. Ask for them as "The readings for CBI Unit 225." Do **not** ask for them by book title.

MODEL EXAM

$$R_\pi = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$R_\sigma = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$$

$$T_\pi = \frac{2 \sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)\,\cos(\theta_i - \theta_t)}$$

$$T_\sigma = \frac{2 \sin\theta_t \cos\theta_i}{\sin(\theta_t + \theta_i)}$$

1. See Output Skill K1 in this module's *ID Sheet*.
2. Calculate the Brewster angle for light from air incident on water ($n = 1.33$). [C]
3. For unpolarized light incident from air on the air-water interface at an angle of incidence of 60°, evaluate the angle of refraction in the water (n=1.33) and evaluate the 4 coefficients: R_π, R_σ, T_π, and T_σ. [I]
   a. What does a negative sign for any of these coefficients mean? What does a positive sign mean? [A]
   b. If the incident light is completely polarized in the plane of incidence, what does that say about $\hat{E}$ and $\hat{B}$? [E]
   c. If the incident light is completely polarized perpendicular to the plane of incidence, what does that say about $\hat{E}$ and $\hat{B}$? [D]
   d. With the incident light completely polarized in the plane of incidence, what fraction of the incident light is reflected? [B]
   e. Repeat the calculation of (d) for the case of light polarized perpendicular to the plane of incidence. [H]
   f. If the incident light consists of 75% light polarized in the plane of incidence and 25% polarized perpendicular to the plane of incidence, what fraction of the energy of the incident light is transmitted? [F]

Brief Answers:

A. See the definition of the reflection and transmission coefficients. If the sign is negative then reflected E_π, for example, is directed opposite to incident E_π, etc.

B. 0.4%.

C. The angle whose tangent equals 4/3, 52.9°.

D. $\vec{E}$ is completely perpendicular to the plane of incidence, $\vec{B}$ is completely in the plane of incidence.

E. $\vec{E}$ is completely in the plane of incidence, $\vec{B}$ is completely perpendicular to this plane.

F. 0.968.

G. $\approx 52.5°$.

H. 11.4%.

I. $\theta_t = 40.6°$; $R_\pi = +0.07$, $R_\sigma = -0.34$, $T_\pi = 0.70$, $T_\sigma = 0.66$.