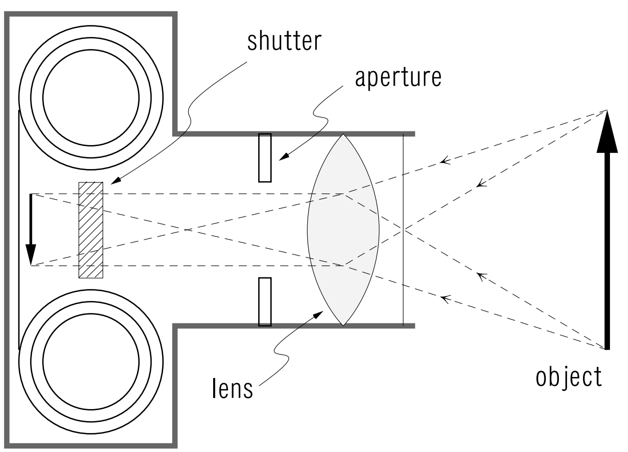
MISN-0-220

LIGHT AS RAYS: SNELL'S LAW



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

LIGHT AS RAYS: SNELL'S LAW

by
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Title: **Light as Rays: Snell's Law**

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Input Skills:

1. Vocabulary: plane wave, spherical wave (MISN-0-203); light wave, photon (MISN-0-212).
2. Basic geometry and trigonometry, including small angle approximations (MISN-0-401).

Output Skills (Knowledge):

K1. Vocabulary: angle of incidence, angle of reflection, angle of refraction, critical angle, geometrical optics, index of refraction, light ray, ray diagram, reflection, refraction, Snell's law, total internal reflection.

K2. State the three rules of light-ray optics.

K3. Derive the condition for total internal reflection.

K4. Use the wave picture to derive Snell's Law.

Output Skills (Rule Application):

R1. Given any two of the following quantities, use the rules of light-ray optics to find the third quantity: the angle of incidence, the angle of refraction, and the relative index of refraction of the media (alternatively, the indexes of refraction of the individual media may be specified so one is given three quantities and is to find the fourth).

R2. Given either the critical angle for total internal reflection or the relative index of refraction of the two media involved, calculate the other quantity (alternatively, the indexes of refraction of the individual media may be specified so one is given two quantities and is to find the third).

Output Skills (Problem Solving):

S1. Given a parallel-faced (planar) slab or a prism, start with the rules of light-ray optics and use them and trigonometry to solve problems involving ray tracing.

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1. Introduction

1a. Sources of Light. There are two types of sources of light: luminous sources and reflective sources. Luminous sources emit light without being illuminated by light from any other source, while reflective sources emit light only while illuminated. Light is emitted, transported, and absorbed in the form of massless particles called photons that carry the light's energy and momentum.

One luminous source is the sun, the photons of light being emitted when atoms and nuclei in the sun undergo energy transitions. Luminous sources on earth include light bulbs, fireflies, and light-emitting diodes (LED's).

For light to be emitted from a reflective object, photons from some other source must be incident upon the surface of the object. These incident photons are absorbed by the atoms and molecules of the reflective object. The energy imparted by the absorbed photons goes into any of a variety of possible excitations in the object. The material in the object very quickly de-excites, emitting photons whose energies (and hence frequencies) are characteristic of the material. Thus a "yellow" object, illuminated with white light, absorbs the white light and then emits yellow light and so looks yellow to an observer.

1b. Wave Properties Dominate. The direction of emission of each photon from a source is not precisely predictable, but the probability for emission in a given direction can be predicted with precision. Hence when very large numbers of photons are involved one can predict with some certainty the direction of emission of the reflected light. This is the case in the phenomena we study in this module and the subject is called Geometrical Optics. It means that we specify geometrical layouts of light sources and materials and then use a little trigonometry to determine how light beams change direction as they pass from one material to the next or when they enter or leave a vacuum.

1c. Light Speed Depends on Medium. When light travels through a substance, such as glass, the explanation for the behavior of the transmitted light is again simple when very large numbers of photons are involved so you can treat the incident and transmitted light as waves. For such cases, the transmitted wave behaves as if the speed of the light wave in the medium is less than c, the speed of light in vacuum. The speed of the wave in the medium is characteristic of the material, meaning it is different for different materials. However, this lowered speed for the wave does not mean that an individual photon in the beam is slowed down in the medium; a photon always travels at the speed c. What does happen is that the photons in the beam are repeatedly absorbed and reemitted by the molecular structure of the medium, with the collective effect being that the wave travels through the medium with an effective speed less than c. For example, the effective speed of light in water is about three-fourths its speed in vacuum, while in some plastics it is less than half that speed.

1d. Geometrical Optics: Tracing Light Rays. Geometrical Optics prepares you to predict or design the paths of light rays as they traverse various solids, liquids, and gases. The materials through which light rays pass are referred to in Geometrical Optics as "media." It is the task of Geometrical Optics to accurately predict and explain the ray paths, and to provide the means to design technological devices that use light rays traversing media.

1e. Bending of Rays in Non-Homogeneous Media. When a light ray traverses a homogeneous medium its path is a straight line, but when it traverses a medium with a density gradient the light ray tends to bend toward that gradient. These reactions, to both homogeneous and non-homogeneous media, form the basis for understanding rainbows, mirages, and the human eye, and for designing lenses for eyeglasses, telescopes, microscopes, and high-tech optical circuits.

1f. Gradual vs. Sudden Changes in the Media. As they speed along, light rays sometimes encounter small density gradients that persist over long paths, as with temperature gradients in air that cause mirages: in such a case the path gradually curves toward the region of higher density (lower temperature in air). By way of contrast, in most optical devices the light ray encounters a sudden change as it passes, say, from air to glass or from water to air: in these cases the ray will undergo a sudden change in direction due to the sudden interaction with the gradient at the interface of the two media.

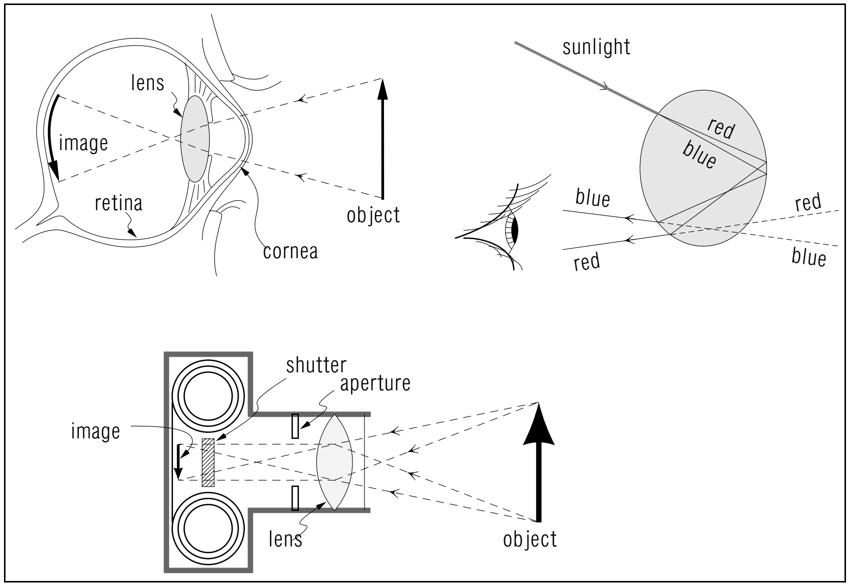

Figure 1. Ray tracing, source to detector: human eye, rainbow-producing raindrop, camera.

2. Light Rays

2a. Source to Detector. In Geometrical Optics we are normally interested in the paths of light rays as they move from a source to a detector. The source might be luminous, such as the sun, a light bulb, or a light-emitting diode (LED). On the other hand, the source might be an illuminated object, such as any object you see by reflected light (a desk, a person, the sky, blades of grass). The detector might be a person's eye, or film in a camera, or a photocell in a video camera or in a remote-sensing satellite.

2b. Examples. In Fig. 1 we show some examples of rays going from sources to detectors. In each case the only rays in which we are interested originate in a common source and wind up in a common detector. We are interested in multiple rays from source to detector because we are interested in forming complex images. By way of contrast, for optical circuits, where light beams are used in place of electric currents, we are generally interested in only single rays going from sources to detectors because there we are only interested in an on-off toggling of intensity.

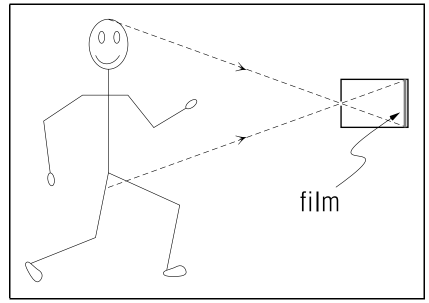

Figure 2. Ray tracing for a pin-hole camera.

2c. Example of Image Formation. To illustrate image formation we analyze how a simple pin-hole camera works. Real cameras become pin-hole cameras when you use a very large f-stop, corresponding to a very small aperture for admitting light to the inside of the camera.

In Fig. 2 the person is the source of the light rays and the sheet of film in the back of the camera is the detector. The front of the camera contains a tiny hole (a "pin-hole"), so the entire paths of the rays shown are in air. The air is homogeneous, so the light rays travel in straight lines. We are free to draw light rays coming out of the source in any direction, but we only draw useful ones. Here, we only draw those heading toward the camera since those are the only rays that have a chance of being recorded on the film.

We wish to predict what parts of the figure will be recorded on the film as an image. For this purpose we need only draw the two rays shown in the figure. It is immediately apparent that: (1) the image will be upside down; and (2) that the part of the person below the knees will not be in the image on the film (draw a straight line from the person's foot, going through the pin-hole: it will not strike the film). Of course the camera could be tilted downward slightly, thereby capturing an image of the whole person.

2d. Wave Description of Rays. A light ray is actually the trajectory of a point on a light wave, as illustrated in Fig. 3. Here the ray is the path followed by the point P, which happens to be on the crest of a wave. As the crest moves along at the speed of light, the point moves along with it.

Succeeding waves are emitted from the source at a constant frequency, as seen in the figure. Because the waves move out at a constant velocity, they don't pile up anywhere. Because they don't pile up anywhere, the frequency of their emission is the same as the frequency with which they

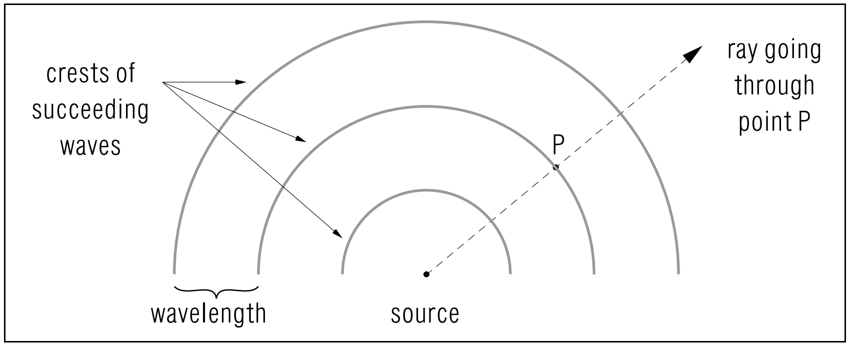

Figure 3. A ray as the trajectory of point P.

pass any stationary point in space. The velocity of the wave, divided by this frequency, is the distance between successive waves, the "wavelength" in this medium (air).

The successive waves are identical, except for their times of emission, so we see that the various crests show where P was at certain times in the past and where it will be at certain times in the future. All of those past and future points are on the ray going through point P (see the figure). The ray is perpendicular to each crest it crosses and is always tangent to a wave's velocity vector as the wave moves along. One can think of the ray as stationary and of the wave crests as following the ray much as a train moves on a track.

2e. Example of Ray Bending: Mirages. Figure 4 shows the path of the light rays that produce the commonly observed highway mirages. The rays come from the sky and are initially heading for the pavement. However, the air near the pavement is hotter than the air above it and

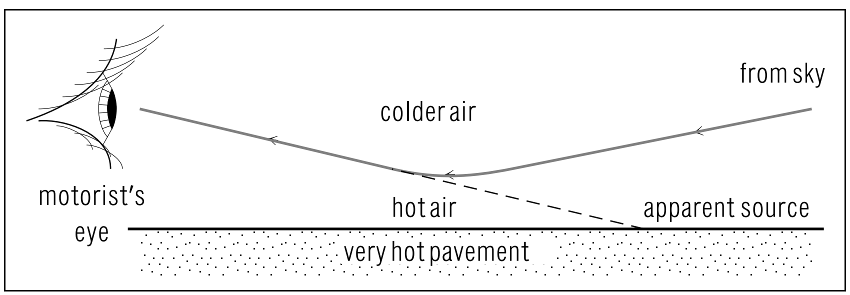

Figure 4. Temperature gradient bends a light ray.

hence is less dense. Thus the lower part of wave goes faster than the upper part of the wave. One can think of the lower part of the wave as getting slightly ahead of the upper part, causing the wave to change its direction (see the figure). The observer can look slightly downward, toward the pavement up ahead, and see light rays that are the exact color of the sky since that is where they originated. This appearance of the sky in the pavement is the mirage. There are other types of mirages, and all of them occur because light waves bend toward the direction of density gradients in the atmosphere. Mirages can be created in the student physics laboratory.

3. Refraction

3a. Refraction of Rays. The bending of a light ray due to the existence of a speed-of-light gradient is called *refraction*. Although mirages are caused by a weak vertical speed-of-light gradient, most practical cases of refraction are caused by a sudden change in speed as a light wave suddenly exits one medium and enters another. This is illustrated in Fig. 5 where you can see that an abrupt change of speed at the interface between the media causes an abrupt change in the direction of the wave.

3b. Derivation of Snell's Law. Snell's Law describes, mathematically, the abrupt change of direction of a light ray as it crosses a planar boundary between two different media. This is illustrated in Fig. 5, where a beam of light goes from a sparser medium, for example air, to a denser medium, for example glass. The angle between the incident ray and the normal to the boundary plane is called the *angle of incidence* and is denoted θ_1. The angle between the refracted ray and the normal to the boundary plane is called the *angle of refraction* and is denoted θ_2 (see the upper part of Fig. 5). *Help: [S-8]*).

Three successive wave peaks and two useful ray lines are shown in the figure. The speeds of light in the two media are designated v_1 and v_2, where v_2 is obviously smaller than v_1.

Now note that the left part of a wave peak reaches the interface before the right part of the same wave peak, so the left side slows down while the right side is still flying along in the sparse medium.

The lower part of Fig. 5 is an enlarged snapshot of the interface region taken at a time t after the left part crosses the interface, when the left part of the wave has traveled a distance Δ_2 rather than the Δ_1 it would have traveled if the interface had not been there (see the figure).

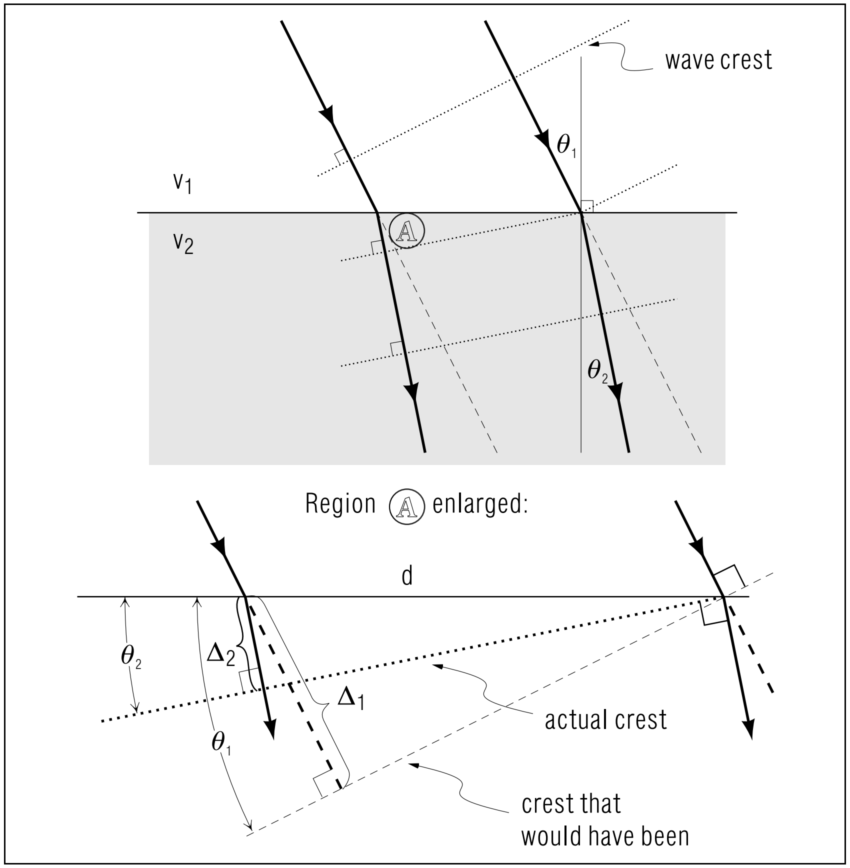

Figure 5. Wave and ray change direction at a media interface, illustrated for passage to a denser medium. The angle of incidence is labeled θ_1. The angle of refraction if labeled θ_2. The distances Δ_1 and Δ_2 are used in the discussion in the text.

▷ Why is the angle labeled θ_1 in the upper part of the figure the same size as the angle labeled the same way in the lower part of the figure? *Help: [S-9]*

Now by trigonometry:

$$\sin\theta_1 = \Delta_1/d,$$

$$\sin\theta_2 = \Delta_2/d.$$

The time t of the snapshot is related to these two distances by the speeds in the two media:

$$t = \Delta_1/v_1 = \Delta_2/v_2.$$

Substituting Δ_1 and Δ_2 from the previous two equations, and canceling the d's, we get:

$$(1/v_1)\sin\theta_1 = (1/v_2)\sin\theta_2.$$

Finally, we define a medium's *index of refraction*, n, as the ratio of the speed of light in vacuum to the speed of light in the medium:

$$n \equiv c/v.$$

Multiplying both sides of our equation by c, the speed of light in vacuum, and substituting n for c/v, we then have Snell's Law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

Snell's Law contains four quantities in one equation so, given any three of the quantities, you can solve for the fourth.

▷ Look back at Fig. 5 and mentally replace the symbols v_1 and v_2 in the upper part of the figure with their equivalent quantities n_1 and n_2.

Sometimes a *relative index of refraction* is defined: $n_{21} \equiv n_2/n_1$. With it, Snell's Law becomes a relationship between three quantities:

$$n_{21} = \sin\theta_1/\sin\theta_2.$$

Some typical values for n are:

medium:	n:
air	1.0003
water	1.333
glass	1.5-1.9

3c. The Wave Explanation for Refraction. It is important to gain an intuitive grasp of the physics of refraction. Start by making sure you can describe, qualitatively (without equations), what is happening in Fig. 5. Extend the discussion to the reverse process that occurs when the light ray inside the medium reaches the other side and exits back into the air. The part of the wave that leaves the glass first speeds up, causing the wave to bend away from the normal to the interface. This phenomenon is illustrated in Fig. 6 for three objects.

▷ Use changes in wave speeds at interfaces (as in Fig. 5) to explain, qualitatively, all ten changes in wave directions shown in Fig. 6.

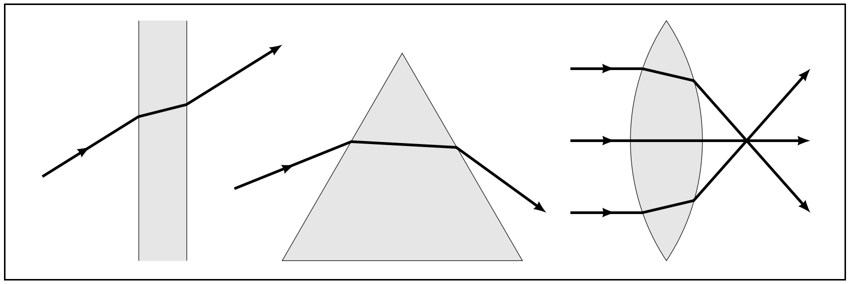

Figure 6. Rays traversing a slab, a prism, a lens.

4. Reflection

4a. Ordinary Reflection. When a ray of light meets an interface between two media, part of the ray is refracted and the rest is *reflected*, as shown in Fig. 7. A perfectly black object reflects none of the incident light, while a perfect mirror reflects all of it. Other materials are somewhere in between. For example, there is very little difference in the color of skin between the various human races, but large differences in *reflectivity* (the fraction of incident light that is reflected).

The angle between a reflected ray and the normal to an inter-media boundary is called the system's *angle of reflection*. For light impinging on a smooth surface the angle of reflection is always equal to the angle of incidence, as shown in the figure. By a "smooth surface" we mean one whose blemishes (departures from flatness) are small in size compared to the wavelength of the light.

4b. Total Internal Reflection. If a ray approaches the surface from inside a denser medium, and the angle of incidence is larger than a particular value, there is no refracted ray: all of the incident light is reflected and there is said to be *total internal reflection*. The angles of incidence that produce this effect are those for which Snell's Law gives values of $\sin\theta$ which are larger than unity,[1] where θ is the angle of refraction. Now $\sin\theta$ cannot be larger than unity, so refractive solutions do not exist for such cases. We suggest that you make a sketch of a prism, showing the refracted rays corresponding to various angles of incidence from inside the prism. Be sure to show cases of total internal reflection. Total internal reflection makes a perfect mirror surface, but you cannot see yourself in this "mirror."

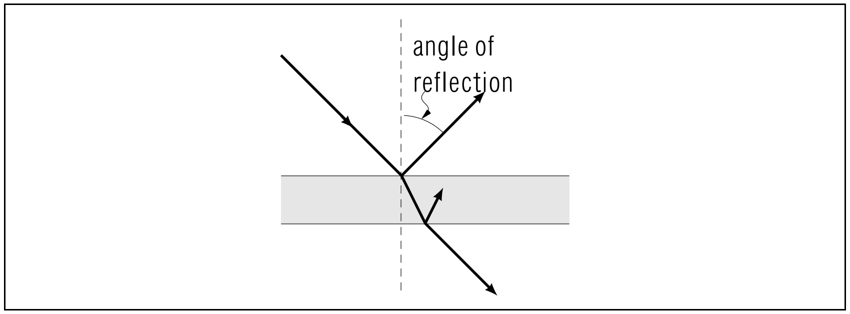

Figure 7. Reflections at media interfaces.

The phenomenon of total internal reflection makes the rainbow possible, since a total internal reflection occurs inside each water droplet that participates in producing the rainbow (see Fig. 1).

5. Summary: Rules of Light-Ray Optics

All problems in ray-tracing can be solved through judicious application of these three rules, plus trigonometry:

1. The incident, reflected, and refracted rays, and the boundary-surface normal, all lie in the same plane.
2. The incident and reflected rays make the same angle to the boundary-surface normal.
3. The incident and refracted rays have the same value for $(n\sin\theta)$: $n\sin\theta = n'\sin\theta'$.

Acknowledgments

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Glossary

- **Angle of incidence**: the angle an incident ray makes with the normal to the boundary surface upon which it is incident.
- **Angle of reflection**: the angle a reflected ray makes with the normal to the boundary surface at the point where the reflected ray originates.

[1] The word "unity" means the number one ("1").

- **Angle of refraction**: the angle a refracted ray makes with the normal to the boundary surface at the point where the refracted ray originates.
- **Critical angle**: the minimum angle of the incident ray at which there is total internal reflection (no refracted ray). This occurs where the sine of the incident angle is the ratio of the two indices of refraction.
- **Geometrical optics**: the use of indexes of refraction and their geometrical boundaries to determine the paths of light rays.
- **Index of refraction**: the property of a homogeneous medium that is the speed of light in vacuum divided by the effective speed of light in the medium.
- **Law of reflection**: for light beams, the equality of the incident and reflected angles.
- **Law of refraction (Snell's Law)**: that the product of the index of refraction and the sine of the incident angle equals the same product for the refracted angle (each with its own index of refraction).
- **Light ray**: the trajectory followed by a point on a light wave. The trajectory is continuous and everywhere tangent to the velocity vector of the wave.
- **Ray diagram**: a diagram showing light rays which represent the motions of light waves that are all in phase. Only selected light rays are shown on such a diagram, since the number of actual rays that can be drawn is infinite.
- **Reflection**: the phenomenon of a light ray changing its direction as it traverses a medium due to collision with the surface separating its medium and another (somewhat the way a ball bounces off a surface between air and solid ground).
- **Refraction**: the phenomenon of a light ray changing its direction as it passes from one medium into another.
- **Snell's law**: the law governing the change of directon of a light ray undergoing refraction: $n \sin\theta = n' \sin\theta'$, where n is the index of refraction for the medium of the incoming part of the ray's path and θ is the angle between the incident ray and the normal to the surface at the point of incidence, and where the primed quantities are defined the same way but for the outgoing ray in the second medium.
- **Total internal reflection**: the case where Snell's Law is invalid because $\sin\theta$ for the refracted ray would be larger than unity. For this case there can be no refracted ray and the intensity of the reflected light equals the intensity of the incident light.

- **unity**: the number "one" (1).

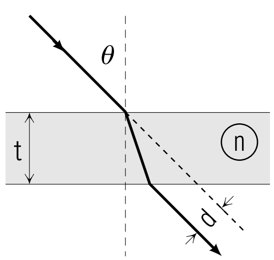
PROBLEM SUPPLEMENT

Note: Problems 1 and 5 also occur in this module's *Model Exam.*

Advice:

- Sketch the prisms, slabs, or other objects containing the relevant interfaces.
- Sketch-in the ray path, roughly to scale.
- Label all angles and dimensions.
- Sketch-in the normal to each interface at the point where the ray strikes it.
- For a case where you are at the borderline between total internal reflection and ordinary refraction, so the angle of refraction is 90°, check that the appropriate part of the ray is indeed along the interface.

1. A light ray traverses a planar slab of glass (one with parallel sides), as shown in the sketch.



   a. Use Snell's Law at each surface to determine d/t, where d is the the lateral displacement of the ray and t is the thickness of the slab. The answer is to be a function of n, the index of refraction of the glass, and θ, the angle of incidence of the ray. Note: $\sin(\theta-\theta') \equiv \sin\theta\cos\theta' - \cos\theta\sin\theta'$. *Help: [S-1]*

   b. determine d: $n = 1.6$, $t = 8.0\,\text{cm}$, $\theta = 45°$.

   Use the answer to part (a) to determine d/t for all five cases below and sketch the path of the ray for the first four cases:

   c. $n \to 1$,

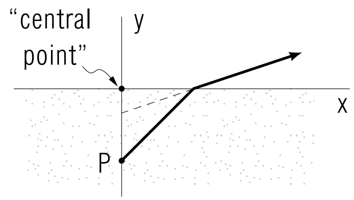
   d. $n \to \infty$,

   e. $\theta \to 0$,

   f. $\theta \to \pi/2$,

   g. θ becomes very small but not zero. *Help: [S-8]*

2. A point source of light, labeled P in the sketch, is at a depth of 10.0 m below the surface of the water ($n = 1.33$). We define a "central point" as being on the surface of the water directly above the light source. The dashed line in the sketch is a projection backwards of the ray shown after it emerges from the water (thus it "appears" to have come from a shallower depth than P).

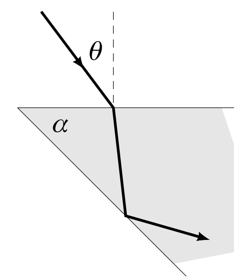


   a. Draw a diagram of the situation.

   a. Determine the farthest distance from the central point where a ray can emerge. *Help: [S-5]*

   b. For an observer who sees the ray emerging 5.0 m from the central point, determine the apparent depth of the source (trace back along the ray to the vertical line above the actual source, as shown in the sketch). *Help: [S-3]*

3. A light ray enters and traverses the apex of a prism. The angle of the apex is α (see sketch) and the index of refraction of the prism is n.



   a. Determine the minimum angle of incidence, θ, such that the ray will be totally reflected internally when it reaches the far side of the apex (see the sketch, which does not necessarily show the minimum angle).

   b. Calculate the angle in part (a) if the prism has an index of refraction of 1.60 and an apex angle of 16.6°.

   c. Sketch the situation where the angle of incidence, θ, is zero and there is just barely total internal reflection. Determine the index of refraction, for this case, as a function of α.

   d. Determine the angle of part (a) when the index of refraction is unity.

4. Suppose we have constructed an experimental set-up in which light rays make an angle of 45° with the face of a prism having a 60° apex. The light rays cross this apex and emerge from the other side. Their emerging angles can be measured and from this value the index of refraction can be measured. Using, in succession, three individual pure-color rays, the index of refraction of the prism is found to be:

color:	wavelength(nm):	n:
blue	475	1.463
yellow-green	550	1.460
red	675	1.456

The three rays are now combined into a single ray which, without the prism, appears to the eye to have the color "white." Sketch what happens as this "white" ray approaches, traverses, then emerges from the prism. Exaggerate the path differences sufficiently to show what happens.

5. A ray of light travels from air into glass with an angle of incidence θ. It is observed that the angle between the reflected and refracted rays is 90°.
   a. Sketch a ray diagram showing the incident, reflected and refracted rays at the interface.
   b. Determine the index of refraction (relative to air) of the glass. Your answer will be a function of θ.
   c. For a ray going from the glass to air what is the maximum angle of incidence, in the glass, for which there will be a refracted ray in the air?
   d. For angles greater than the angle determined in part (c), if there is no refracted ray emerging into air what happens to the light that is incident upon the interface?

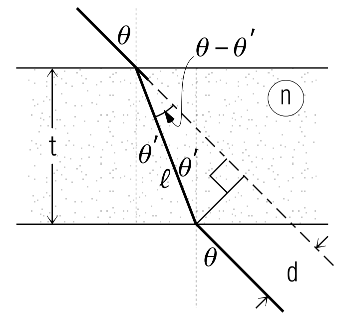
6. (Only for those interested) A ray of light is incident at an angle of incidence θ upon a glass plate of thickness A and parallel sides.
   a. Prove that the ray entering the glass and emerging from the other side is parallel to the incident ray.
   b. Verify that the lateral displacement of the emergent ray is:

$$D = A\frac{\sin(\theta - \phi)}{\cos\phi}$$

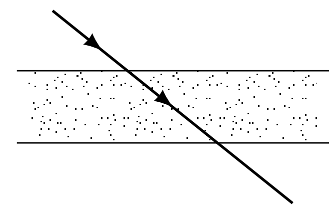
   where ϕ is the angle of refraction of the ray in the glass and θ is its angle of incidence.

Brief Answers:

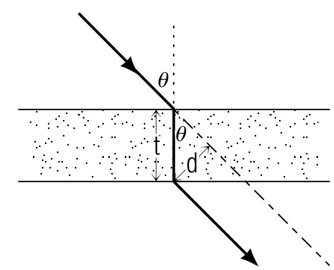
1. a. By trigonometry in the sketch:
   (i)$d/\ell = \sin(\theta - \theta')$; *Help: [S-1]*
   (ii)$t/\ell = \cos\theta'$; *Help: [S-1]*
   and by Snell's law in the sketch:
   (iii) $\sin\theta = n\sin\theta'$
   Using the trigonometric identity
   $\sin(\theta - \theta') = \sin\theta\cos\theta' - \cos\theta\sin\theta'$;
   we get:
   $d/t = \sin\theta[1 - \cos\theta/(n\cos\theta')]$
   where:
   $\cos\theta' = \sqrt{1 - (\sin^2\theta)/n^2}$



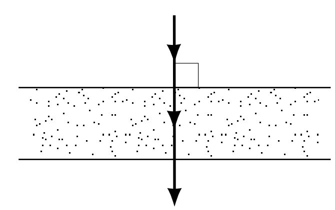
   b. Angle of refraction is 26.2°, lateral displacement is 2.87 centimeters. *Help: [S-2]*
   c. $d/t \to 0$



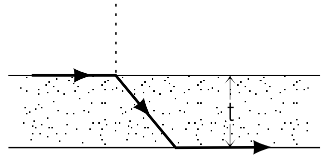
   d. $d/t \to \sin\theta$



   e. $d/t \to 0$



f. $d/t \to 1$



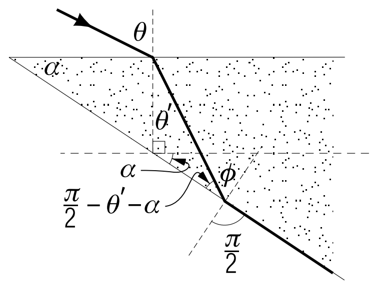
g. $d/t \to \theta[(n-1)/n]$ *Help: [S-11]*

2. a. 11.4 m. *Help: [S-2]*

   b. 6.8 m. *Help: [S-3]*

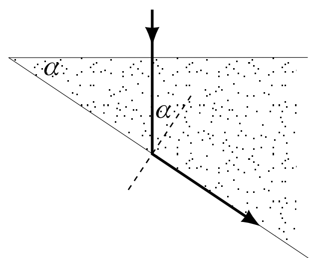
3. *Help: [S-4]*

   a. $\sin\theta = n\sin\theta'$
   $n\sin\phi = \sin(\pi/2) = 1$
   where: $\phi = \ldots = \theta' + \alpha$.
   Solving, we get for the minimum angle of incidence: $\theta = \sin^{-1}\{n\sin[-\alpha + \sin^{-1}(1/n)]\}$

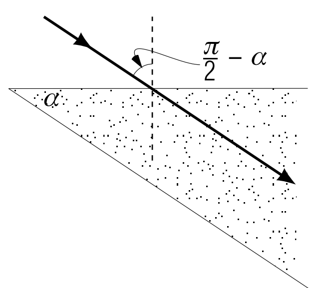


   b. $\theta = 37.0°$.

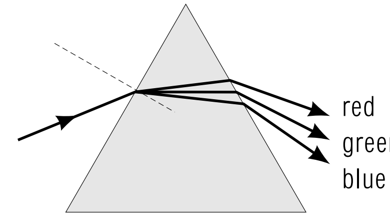
   c. By inspection of the last equation above, $n = 1/\sin\alpha$ does the job (plug it in and see it work!) or simply apply Snell's law at the point of total internal reflection.



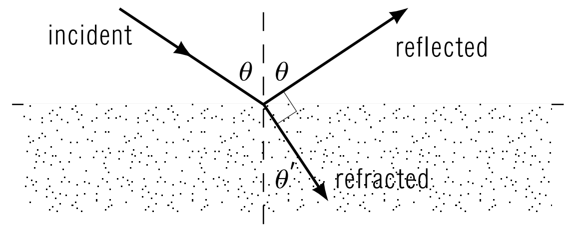
   d. $\theta = \pi/2 - \alpha$.



4. Here is a sketch, not to scale. This shows how a prism produces a rainbow-like spectrum from white light.



5. a.



   b. $n = \tan\theta$. *Help: [S-9]*

   c. ϕ is the maximum angle, where $\sin\phi = \cot\theta$, θ being the angle referred to in part (b).

   d. It is totally reflected, back into the medium from which light was incident upon the interface.

6. (proofs)

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-Problem 1a)*

1. Copy the ray diagram given in the problem.
2. Label the incident and refracted rays at the air-glass interface first encountered by the ray.
3. Where the incident ray approaches the glass, label the angle of incidence θ.
4. Where the refracted ray enters the glass, label the angle of refraction θ'.
5. Extend the incident ray, as a dashed line, along the path the ray would have taken had the glass not been there.
6. Note that the dashed line is parallel to the actual ray after the latter emerges from the glass.
7. Draw a line that: (1) starts from the point where the actual ray emerges from the glass; (2) ends on the dashed line; and (3) is perpendicular to the dashed line. Mark this line's length as d.
8. Note that d is the lateral displacement we are looking for.
9. Label the ray's actual path length inside the glass ℓ.
10. Mark the angle between the refracted ray and the dashed line as: $\theta - \theta'$. Note that $\sin(\theta - \theta') = d/\ell$.
11. Draw a line that: (1) starts from the point where the ray first enters the glass; (2) ends on the final glass surface; and (3) is perpendicular to that surface. Label the length of this line t, the thickness of the glass.
12. Note that $\cos\theta' = t/\ell$.
13. Substitute for $\sin(\theta - \theta')$ using the trig identity quoted.
14. Use Snell's law for θ'. Solve for d in terms of t, θ, n. *Help: [S-7]*

S-2 *(from PS-Problem 1b)*

If you calculate using angles in degrees, you will only get the right answer if your calculator is set for degrees, not radians, not grads.

S-3 *(from PS-Problem 2c)*

$5.0\,\mathrm{m}/\tan[\sin^{-1}\{1.33\sin(\tan^{-1}|5.0\,\mathrm{m}/10.0\,\mathrm{m}|)\}] = 6.8\,\mathrm{m}$ *Help: [S-12]*

S-4 *(from PS-Problem 3)*

Get all the right answers to problems 1 and 2 before attempting this problem. Then:

1. Draw a light ray entering a prism, traveling through it, and emerging from it. Draw a perpendicular to the surface at the points where the ray enters and leaves the prism. Make sure your ray (roughly) obeys Snell's law, bending toward the perpendicular to the surface when appropriate and bending away from it when appropriate.
2. Now draw another ray entering the prism at the same spot but with a different angle of incidence. Follow the ray through the prism and out into the air.
3. Now draw a ray that winds up moving *along* the second interface (a 90 degree angle of refraction at the second interface). This ray is on the border between being refracted out into the air and being totally reflected internally. This is the totally reflected ray with the minimum angle of incidence.

S-5 *(from PS-Problem 2)*

Get all the right answers to problem 1 before attempting this problem.

S-6 *(from PS-Problem 2a)*

This problem involves total internal reflection. Also, see the advice given at the beginning of the *Problem Supplement*.

S-7 *(from [S-1])*

[S-1] gives:

$$(d/t) = \sin\theta - \cos\theta \tan\theta' ,$$

and Snell's law at either interface gives:

$$\theta' = \sin^{-1}\left(\frac{\sin\theta}{n}\right).$$

Together, the above two equations constitute a quite acceptable answer for d/t as a function of n and θ since one can first calculate θ' and then substitute it into the equation for d/t.
An equivalent solution can be constructed by substituting Snell's law,

$$\sin\theta' = \frac{\sin\theta}{n},$$

into the definition of the tangent:

$$\tan\theta' = \frac{\sin\theta'}{\cos\theta'} = \frac{\sin\theta}{n\cos\theta'},$$

and then substituting the latter expression back into the relation for d/t.
Here is how we obtained the answer given in the Brief Answers: Imagine an angle θ' in a right-triangle angle with opposite side $\sin\theta$ and hypotenuse n so Snell's law is obeyed. Then the adjacent side is $\sqrt{n^2 - \sin^2\theta}$ and:

$$\cos\theta' = \frac{\sqrt{n^2 - \sin^2\theta}}{n} = \sqrt{1 - \frac{\sin^2\theta}{n^2}}.$$

S-8 *(from TX, Sect. 3b)*

Note that the two angles labeled θ_1 have the same values because they each represent angles between lines that are 90°apart (note the "right angle" marks on the right side of the lower part of the figure).

S-9 *(from PS-Prob. 4b)*

In the denominator we have: $\sin(180° - \theta - 90°) = \sin(90° - \theta) = \cos(\theta)$. That last equality is apparent by considering these relationships in a right-angle triangle: (i) if one acute angle is (θ), the other is $(90° - \theta)$; and (ii) the sine of an acute angle is the opposite side over the hypotenuse while the cosine is the adjacent side over the hypotenuse; so that (iii) the sine of one acute angle is the cosine of the other acute angle.

S-10 *(from TX-Sect. 3b)*

Imagine a rigid counterclockwise rotation of the (initially horizontal and vertical) axes about the point where 5 lines intersect at the right side of the lower part of Fig. 5. Rotate the axes through the angle θ_1, imagining that the axes are parts of a single rigid object. Being rigidly connected, they must both rotate through the same angle. Thus the two angles marked θ_1 must have the same values.

S-11 *(from PS-Prob. 1g)*

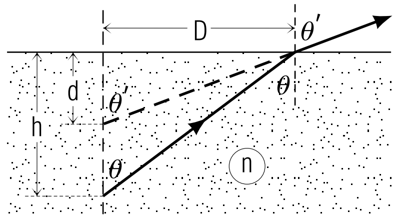
For small angles ($\theta \ll 1$ radian) these substitutions can usually be made: $\sin\theta \Rightarrow \theta$, $\tan\theta \Rightarrow \theta$, $\cos\theta \Rightarrow 1$.
This means that, for small angles, you can replace the sine of the angle by the angle itself, etc. The smaller the angle, the less *fractional* error is incurred by making such replacements.

θ(rad)	$\sin\theta$	error (%)
1.00000	0.84147	15.8529
0.10000	0.09983	0.1666
0.01000	0.01000	0.0017

For more insight, and for a discussion of when the substitution must be made more subtly, see "Taylor's Polynomial Approximation for Functions", MISN-0-4.

S-12 *(from Help: [S-3])*

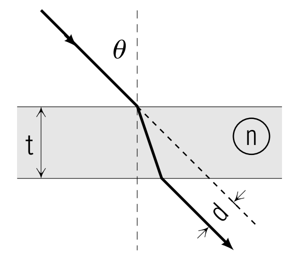
$$n \sin\theta = \sin\theta'$$
$$\tan\theta' = \frac{D}{d}$$
$$\tan\theta = \frac{D}{h}$$



Combine those to eliminate θ and θ', and the answer will contain tan, $\sin^{-1}$, sin, and $\tan^{-1}$. As a check, note that if $n = 1$ (water replaced by air), sin cancels $\sin^{-1}$ and then tan cancels $\tan^{-1}$ and $d = h$ as it should. You can also see the check in the three equations above.

MODEL EXAM

1. See Output Skills K1-K4 in this module's *ID Sheet*.
2. A light ray traverses a planar slab of glass (one with parallel sides), as shown in the sketch.



   a. Use Snell's Law at each surface to determine the lateral displacement of the ray, d, as a function of the index of refraction of the glass, n, the thickness of the slab, t, and the angle of incidence of the ray. Note: $\sin(\theta - \theta') = \sin\theta\cos\theta' - \cos\theta\sin\theta'$.
   b. determine d: $n = 1.6$, $t = 8.0\,\text{cm}$, $\theta = 45°$.

c. Use the answer to part (a) to determine d/t and sketch the path of the ray for the case: $n \to 1$.

3. A ray of light is incident in air upon a glass surface at an angle of incidence θ. It is observed that the angle between the reflected and refracted rays is 90°.
   a. Sketch a ray diagram showing the incident, reflected and refracted rays at the interface.
   b. Determine the index of refraction (relative to air) of the glass. Your answer will be a function of θ.
   c. For a ray going from the glass to air what is the maximum angle of incidence, in the glass, for which there will be a refracted ray in the air?
   d. For angles greater than the angle determined in part (c), if there is no refracted ray emerging into air what happens to the light that is incident upon the interface?

Brief Answers:

1. See this module's *text*.
2. See Problem 1 in this module's *Problem Supplement*.
3. See Problem 5 in this module's *Problem Supplement*.