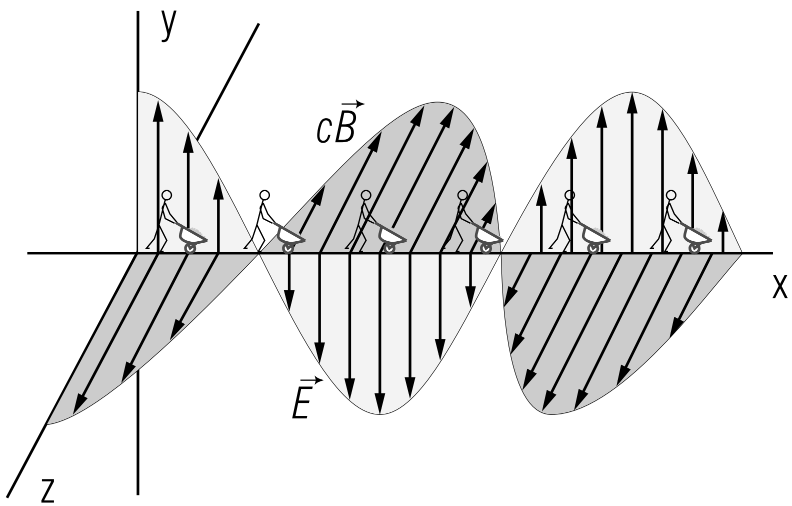
MISN-0-211

ENERGY AND MOMENTUM IN ELECTROMAGNETIC WAVES



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

ENERGY AND MOMENTUM IN ELECTROMAGNETIC WAVES

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Title: **Energy and Momentum in Electromagnetic Waves**

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Input Skills:

1. Vocabulary: electromagnetic wave (MISN-0-210), charge density (MISN-0-147), current density (MISN-0-144).
2. Describe the storage of energy in a capacitor (MISN-0-135).
3. Describe the storage of energy in an inductor (MISN-0-144).
4. Relate the electromagnetic wave's electric and magnetic fields to each other and to the direction of propagation of the wave. (MISN-0-210).
5. Relate force to rate of change of momentum (MISN-0-15).

Output Skills (Knowledge):

K1. Vocabulary: electric energy current density, magnetic energy current density, energy current density, momentum current density.

K2. Given the energy density in an electric fields, and the knowledge that the electric field energy is given by $E_e = (1/8\pi k_e)E^2$ where E is the electric field, derive the expression for the time-average energy current density in the wave.

Output Skills (Rule Application):

R1. Relate the electric and magnetic field amplitudes to the intensity, energy density, and momentum density associated with an electromagnetic wave of a given frequency.

Post-Options:

1. "Brewster's Law and Polarization" (MISN-0-225).

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1. Description

Electromagnetic waves travel at the speed of light, and they transport energy and momentum from the places where they originate to the places they travel to. The energy and momentum they carry can be easily computed from a knowledge of their amplitudes. Knowing the energy carried by a wave, one can calculate the heating produced when an electromagnetic wave is absorbed by a material object. Similarly, knowing the momentum being carried by a wave allows us to calculate the pressure exerted on an object that absorbs, deflects, or reflects the wave.

2. Static Fields

2a. Energy in Static Fields. Elsewhere we have shown that the energy stored in the electric field of a capacitor is:[1]

$$E_e = \frac{1}{2}CV^2 . \quad (1)$$

This amount of energy can be recovered if the capacitor is discharged so its voltage and hence its internal electric field drops to zero.

Similarly, the energy stored in the magnetic field of an inductor is:[2]

$$E_m = \frac{1}{2}LI^2 , \quad (2)$$

and this amount of energy can be recovered if the current through the inductor is shut off so its magnetic field drops to zero.

A crucial point is that one can prove in general that the energy stored in electric and magnetic fields, including the examples shown above, can

[1]See "Capacitance and Capacitors," (MISN-0-135).
[2]See "Inductance and Inductors," (MISN-0-144).

be calculated from:[3]

$$E_e = \int_{\text{all space}} \left(\frac{1}{8\pi k_e}\right) E^2 dV , \quad (3)$$

where E is the electric field, and

$$E_m = \int_{\text{all space}} \left(\frac{1}{8\pi k_m}\right) B^2 dV , \quad (4)$$

where B is the magnetic field.

2b. Energy Density in Static Fields. The integrands in the right hand sides of Eqns. (3) and (4) are over all space, so their integrands can be interpreted as energy densities, energy per unit volume:

$$\frac{dE_e}{dV} = \frac{1}{8\pi k_e}E^2 , \quad (5)$$

and

$$\frac{dE_m}{dV} = \frac{1}{8\pi k_m}B^2 . \quad (6)$$

3. Electromagnetic Waves

3a. Energy Density. For the case of an electromagnetic wave the electric and magnetic amplitudes are related:[4]

$$E = cB ,$$

so we can use that and the relation between the electric and magnetic force constants:[5]

$$k_m = k_e/c^2$$

to rewrite Eq. (6) as the magnetic energy density in an electromagnetic wave:

$$\frac{dE_m}{dV} = \frac{1}{8\pi k_e}E^2 . \quad (7)$$

Note that this is exactly the same as the electric energy density, Eq. (5), so the total energy density for the electromagnetic wave is:

$$\frac{dE}{dV} = \frac{1}{4\pi k_e}E^2 . \quad (8)$$

[3]See any advanced *Electricity and Magnetism* textbook.
[4]See "Electromagnetic Waves From Maxwell's Equations," MISN-0-110.
[5]See "Magnetic Interactions," (MISN-0-124).

3b. Energy Current Density. As an electromagnetic wave travels at the speed of light, the energy it carries can be expressed as an *energy current* in analogy with electric current. Recall that electric current density $\vec{j}$ is defined as the amount of charge crossing unit area perpendicular to the current direction, per second. This current density can be expressed in terms of its charge density ρ and the drift velocity $\vec{v}$ of the charges:

$$\vec{j} = \rho\vec{v}\,. \tag{9}$$

Similarly, we can define the energy current density $\mathcal{E}$ as the energy per unit area perpendicular to the current direction, per second. Then it is given by the analog of Eq. (9):

$$\vec{\mathcal{E}} = \frac{1}{4\pi k_e}E^2\vec{v}\,. \tag{10}$$

Making use of the directions of the fields, this is usually written:

$$\vec{\mathcal{E}} = \frac{c^2}{4\pi k_e}\vec{E}\times\vec{B}\,. \tag{11}$$

3c. Momentum Current Density. For particles that are traveling at high speeds, speeds not negligible compared to the speed of light, momentum and energy are related by:[6]

$$\vec{p} = \frac{E}{c^2}\vec{v}\,. \tag{12}$$

Applying this equation to our electromagnetic wave, where $v = c$, we get, from Eq. (11), the momentum current density:

$$\vec{\mathcal{P}} = \frac{c}{4\pi k_e}\vec{E}\times\vec{B}\,. \tag{13}$$

3d. Time-Average Quantities. The electric and magnetic amplitudes in the energy and momentum current densities, Eqns. (11) and (13), are oscillating functions of time while in most applications we are interested in time-averaged quantities. The conversion is easy because the time averages that occur in those equations are simply cosine-squared functions of time, and any cosine-squared function has an average value of one-half (0.500). Thus:

$$\overline{\vec{\mathcal{E}}} = \frac{c^2}{8\pi k_e}\vec{E}_0\times\vec{B}_0\,. \tag{14}$$

[6]See "Relativistic Momentum," MISN-0-24.

$$\overline{\vec{\mathcal{P}}} = \frac{c}{8\pi k_e}\vec{E}_0\times\vec{B}_0\,. \tag{15}$$

3e. Pressure From Absorption and Reflection. Since force is change of momentum per unit time, the time-average pressure exerted on a surface by an electromagnetic wave is the change of the wave's time-average momentum current density due to its interaction with the surface:

$$\overline{\vec{P}} = \Delta\overline{\vec{\mathcal{P}}}\,.$$

For the case of absorption of the wave,

$$\overline{\vec{P}} = \overline{\vec{\mathcal{P}}} = \frac{c}{8\pi k_e}\vec{E}_0\times\vec{B}_0\,. \tag{16}$$

For the case of perfect reflection where the wave's momentum is simply reversed in direction:

$$\overline{\vec{P}} = 2\overline{\vec{\mathcal{P}}} = \frac{c}{4\pi k_e}\vec{E}_0\times\vec{B}_0\,. \tag{17}$$

Acknowledgments

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Glossary

- **electric energy current density**: the electric-field energy being transported, normal to a plane, per second per unit area in the plane. The SI unit is watts per square meter.
- **magnetic energy current density**: the magnetic-field energy being transported, normal to a plane, per second per unit area in the plane. The SI unit is watts per square meter.
- **energy current density**: the energy being transported, normal to a plane, per second per unit area in the plane. The SI unit is watts per square meter.
- **momentum current density**: the normal component of the momentum at a plane, per unit area in the plane. The SI unit is kilogram meters per second per square meter.

PROBLEM SUPPLEMENT

Note: Problem 3 also appears in this module's *Model Exam*.

1. The electric field of a plane electromagnetic wave in vacuum is represented by:

   $E_x = 0\,,$

   $E_y = 0.50\,(\mathrm{N/C})\cos\left[2.09\,\mathrm{m}^{-1}(x - ct)\right]\,,$

   $E_z = 0\,.$

   Determine the wave's "intensity," its average energy per unit area per unit time.

2. Solve Problem 1 for the wave represented by:

   $E_x = 0\,,$

   $E_y = 0.50\,(\mathrm{N/C})\cos\left[0.419\,\mathrm{m}^{-1}(x - ct)\right]\,,$

   $E_z = 0.50\,(\mathrm{N/C})\cos\left[0.419\,\mathrm{m}^{-1}(x - ct)\right]\,.$

3. The amplitude of the electric field of an electromagnetic wave is $\sqrt{180\pi}$ volts per meter. The electric field is in the $\hat{x}$-direction, the magnetic field in the $\hat{z}$-direction. Determine:

   a. The average energy current density associated with this wave. [C]
   b. The average momentum current density carried by this wave. [F]
   c. The direction of this momentum. [A]
   d. The energy incident in one day upon a surface oriented in the x-z plane that has an area of $2.7\,\mathrm{m}^2$. [D]
   e. The energy incident upon this surface if the surface is oriented in the x-y plane. [G]
   f. The force exerted on this surface by the radiation if it is completely absorbed by the surface. [B]
   g. The direction of this force. [E]

Brief Answers:

1. 3.32×10^{-4} watts/m^2
2. $6.64 \times 10^{-4}\,\mathrm{W/m^2}$

A. $-\hat{y}$.

B. $6.75 \times 10^{-9}\,\mathrm{N}$.

C. $0.750\,\mathrm{J/(m^2\,s)}$.

D. 1.75×10^{5} joules falls on area.

E. $-\hat{y}$.

F. $0.250 \times 10^{-8}\,\mathrm{J/m^3}$.

G. Zero.

MODEL EXAM

$$\vec{\mathcal{E}} = \frac{c^2}{4\pi k_e}\vec{E}\times\vec{B}\,.$$

1. See Output Skills K1-K2 in this module's *ID Sheet*.

2. The amplitude of the electric field of an electromagnetic wave is $\sqrt{180\pi}$ volts per meter. The electric field is in the $\hat{x}$-direction, the magnetic field in the $\hat{z}$-direction. Determine:

   a. The average energy current density associated with this wave.

   b. The average momentum current density carried by this wave.

   c. The direction of this momentum.

   d. The energy incident in one day upon a surface oriented in the x-z plane that has an area of $2.7\,\mathrm{m}^2$.

   e. The energy incident upon this surface if the surface is oriented in the x-y plane.

   f. The force exerted on this surface by the radiation if it is completely absorbed by the surface.

   g. The direction of this force.

Brief Answers:

1. See this module's *text*.

2. See Problem 3 in this module's *Problem Supplement*,