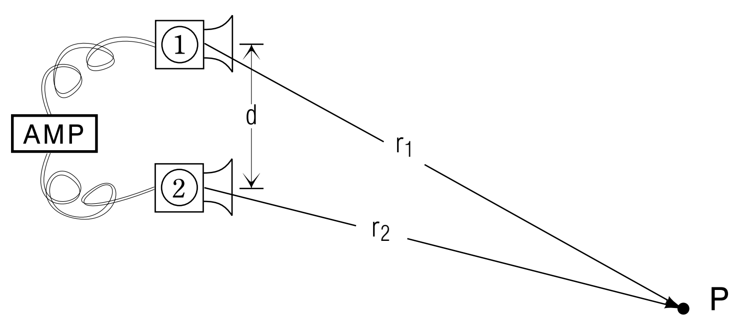
PHYSNET

MISN-0-205

THE INTERFERENCE OF TWO COHERENT WAVE SOURCES



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

THE INTERFERENCE OF TWO COHERENT WAVE SOURCES

by

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Input Skills:

1. Vocabulary: spherical wave, wave front, intensity (MISN-0-203).
2. State the expression for the wave function of a spherical wave (MISN-0-203).

Output Skills (Knowledge):

K1. Vocabulary: central maximum, coherent wave sources, constructive interference, destructive interference, free spectral range, interference pattern, interference order, maxima, minima, path difference, phase difference, phasor.

K2. State the conditions for destructive and constructive interference, both in terms of phase difference and in terms of path difference.

K3. State the expression for the intensity of the net wave disturbance produced by two equal coherent sources, as a function of phase difference.

Output Skills (Problem Solving):

S1. Given two equal coherent wave sources, determine by mathematical and graphical methods the amount of net wave disturbance at a given point in space.

S2. Sketch the wave disturbances, produced at a given point in space by two equal coherent wave sources, as functions of time. Include both the individual disturbances and the net disturbance.

External Resources (Required):

1. Ruler, protractor.

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1. Introduction

1a. Overview. The interference of waves is one of the most important and useful phenomena in physics. Historically, interference effects helped confirm the wave nature of light. The bending of sound and light waves around obstacles and corners, the fundamental limitation of the resolving power of optical instruments, the existence of "dead spots" in an auditorium or a living room, the appearance of colored rings in oil slicks, and the non-reflective properties of thin-metal coatings on lenses are all examples of interference and its related phenomenon, diffraction.

1b. "Coherent" Wave Sources. If two or more wave sources produce waves of the same frequency and wavelength, with the same phase at the same time at their respective origins, the sources are synchronized, or "coherent." As an illustration, think of an infinitely large swimming pool. Out in the middle are two flat discs just under the surface of the water and parallel to the surface. A rod is connected to each disc. The two rods go straight up to an overhead crankshaft that causes the discs to move up and down with the same phase, so they are coherent wave sources. They move up and down with the same amplitude so they are equal sources of waves. Given the frequency and amplitude of these two sources, along with the distance between them, how can we predict and explain the observed wave pattern at all points of the water's surface? How can we do the same for audio waves from multiple sources, or from reflections off walls that can also be considered sources? This unit develops the basic approach for solving such problems.

2. Rotating Phase Vectors

2a. Definition of a Phasor. To geometrically visualize the combination of disturbances from two or more coherent wave sources, we will introduce the idea of an imaginary rotating phase vector, also called a "phasor."

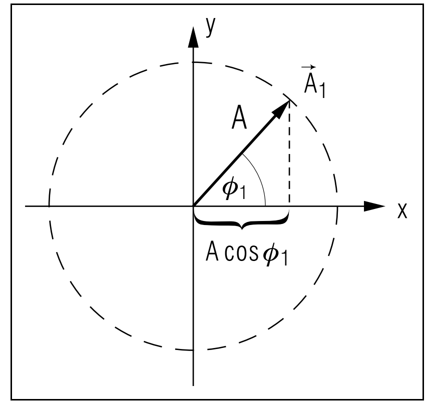


Figure 1. A rotating phase vector and its x-component.

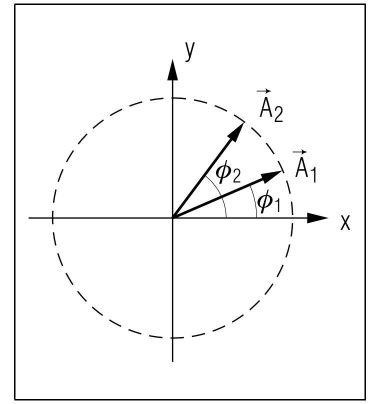


Figure 2. Two phasors of different phase.

Consider a vector $\vec{A}_1$ lying in the x-y plane of a Cartesian coordinate system, as illustrated in Fig. 1. The magnitude of $\vec{A}_1$ is simply A and the vector makes an angle ϕ_1 with the positive x-axis. If $\vec{A}_1$ is made to rotate counterclockwise with angular velocity ω while its tail is fixed at the origin, A_{1x} will vary sinusoidally between values of A and $-A$. The x-component of $\vec{A}_1$ can therefore represent the disturbance produced by a wave at some fixed point in space if the magnitude A is the wave's amplitude, and if ϕ_1 is the phase of the wave disturbance, i.e.

$$A_{1x} = A\cos\phi_1 = A\cos(\omega t + kr_1 + \phi_0)\,,$$

where ω is both the angular frequency of the wave and the angular velocity of the rotating vector, k is the wave number of the wave, and ϕ_0 is the phase constant. The distance from the wave's source to the point of disturbance is r_1. Keep in mind that it is only the x-component of the phasor that is physically meaningful. The y-component does not represent any real quantity.

2b. The Phase Difference of Two Phasors. Now consider the disturbance at the same point in space produced by another wave from a wave source coherent with the first source. The second wave is identical to the first wave, except it travels a different distance r_2. The wave disturbance at the point in question will have a different phase ϕ_2 and be represented by a function $\xi = A\cos(\omega t + kr_2 + \phi_0)$, which can be associated with the x-component of another vector $\vec{A}_2$ as illustrated in Fig. 2. Since the two vectors both rotate counterclockwise at the same angular

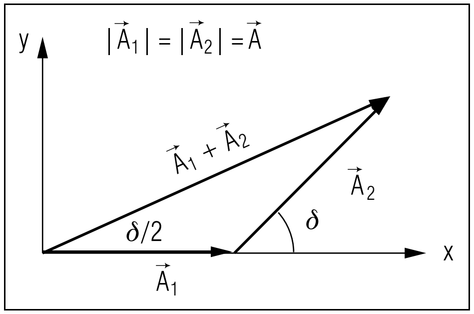


Figure 3. Vector addition of two phasors

rate ω, the angular separation between the phasors remains constant, and is given by:

$$\begin{aligned}\delta &= \phi_2 - \phi_1 \\ &= (\omega t + kr_2 + \phi_0) - (\omega t + kr_1 + \phi_0) \\ &= k(r_2 - r_1) = \frac{2\pi}{\lambda}(r_2 - r_1).\end{aligned}$$

Since δ is the difference between two phases, ϕ_2 and ϕ_1, it is given the name "phase difference," and is independent of time, as seen above. It depends only on the "path difference," $r_2 - r_1$, which is the extra distance that one wave travels relative to the other to reach the point in question. Writing the path difference as Δ, the phase difference δ may be rewritten as:

$$\delta = \left(\frac{\Delta}{\lambda}\right) 2\pi \text{ or } \delta = \left(\frac{\Delta}{\lambda}\right) 360^\circ \tag{1}$$

depending on whether δ is expressed in radians or degrees. The phase difference is interpreted as the fraction or multiple of a complete wave oscillation that one wave has advanced relative to the other. As an example, if λ is 9.0 meters and Δ is 3.0 meters, δ is 120° or $2\pi/3$ radians. We say that the two wave disturbances are "out of phase" by 120° or $2\pi/3$ radians. The fact that δ is independent of time means that $\vec{A}_1$ and $\vec{A}_2$ do not change direction relative to each other and may be combined by vector addition to yield the net result of both wave disturbances.

2c. Vector Sum of the Rotating Phase Vectors. The combination of the disturbances produced by two waves, out of phase by a phase difference δ, can be represented as a problem in the vector addition of two phasors on a phasor diagram. The two phasors illustrated in Fig. 3 represent the disturbance at a fixed point in space produced by two waves of equal amplitude originating from coherent sources. The magnitude of

the phasors represents the amplitude of the waves, and the angle between their directions represents the phase difference δ between the wave disturbances. The entire triangle of phasors is rotating counter-clockwise with angular frequency ω. Fig. 3 is a "snapshot" taken of the diagram when the phasor $\vec{A}_1$ is aligned with the positive x-axis. The vector sum of $\vec{A}_1$ and $\vec{A}_2$, represented by $\vec{A}'$, may be found by resolving the vectors into x- and y-components:

$$\begin{aligned}A'_x &= A_{1x} + A_{2x} = A + A\cos\delta \\ A'_y &= A_{1y} + A_{2y} = 0 + A\sin\delta\end{aligned}$$

The magnitude of A' may be found by applying the Pythagorean theorem:

$$\begin{aligned}A' &= \left[A'^2_x + A'^2_y\right]^{1/2} \\ &= \left[(A + A\cos\delta)^2 + (A\sin\delta)^2\right]^{1/2} \\ &= \left[A^2 + 2A^2\cos\delta + A^2\cos^2\delta + A^2\sin^2\delta\right]^{1/2} \\ &= \left[2A^2 + 2A^2\cos\delta\right]^{1/2}\end{aligned}$$

where use has been made of the fact that: $\cos^2\delta + \sin^2\delta = 1$. We conclude that:

$$A' = \left[2A^2(1 + \cos\delta)\right]^{1/2}. \tag{2}$$

The extension of this method to the addition of wave disturbances from three or more coherent sources is conceptually a simple problem in vector addition, although the algebraic result may not be as concise.[1]

3. Two Coherent Point Sources

3a. Representing the Spherical Waves. Given two coherent point sources, each source will produce a spherical wave of the form:

$$\xi(r,t) = \xi_0(r)\cos(kr - \omega t + \phi).$$

We call these "spherical" waves because the crests and troughs of the waves have the form of concentric outgoing spheres. Of course the waves will become weaker with increasing distance from their sources, but we have neglected this effect in the above equation. We really only care about

[1] The method is used to derive the angular sensitivity of a linear-array radio telescope. See "The Interference of Many Coherent Sources: Radio Interferometry" (MISN-0-231).

the two waves at the receiver, and there the two waves have about the same amplitudes if the receiver is at a large distance from the sources. We will assume that this is true, that the amplitudes from the two sources are equal in magnitude at the receiver, and that interference effects are due to phase differences between the waves.

3b. The Amplitude of the Resulting Wave Disturbance. The amplitude of the net wave disturbance of two spherical waves can be expressed using Eq. (2) if we let $A = \xi_0$ and $A' = \xi_0'$, where ξ_0' is the amplitude of the net wave disturbance. Using another trigonometric identity, $2\cos^2(\delta/2) = 1 + \cos\delta$, Eq. (2) becomes:

$$A' = \left[2A^2\left(2\cos^2\frac{\delta}{2}\right)\right]^{1/2} = \left[4A^2\cos^2\frac{\delta}{2}\right]^{1/2}$$

so,

$$\xi_0' = 2\xi_0\cos\frac{\delta}{2}\,.$$

Thus when $\cos(\delta/2)$ is zero, the amplitude of the net wave disturbance is zero. This situation is called "destructive interference." When $\cos(\delta/2)$ takes its maximum value of ± 1, the net wave disturbance has an amplitude twice the size of either individual wave disturbance. This situation is called "constructive interference." For cases where $0 < |\cos(\delta/2)| < 1$, the interference is neither totally destructive nor totally constructive. Thus the interference pattern consists of variations in the wave disturbance from points where the amplitude is doubled, called "maxima," to points where the amplitude is zero, called "minima."

3c. The Intensity of the Net Wave Disturbance. In a previous module, we have seen that the intensity of a wave is proportional to the square of its amplitude.[2] From Fig. 3 we can see that the net wave disturbance at a given point in space has a phase of $(\omega t + \delta/2)$, assuming that the figure is a snapshot of the rotating phasors at time $t = 0$. Thus the total expression for the net wave disturbance at a given point in space as a function of time is:

$$\xi'(t) = 2\xi_0\cos\left(\frac{\delta}{2}\right)\cos\left(\omega t + \frac{\delta}{2}\right).$$

[2]See "Intensity and Energy in Sound Waves" (MISN-0-203).

The intensity of the wave disturbance as a function of time is therefore expressed as:

$$I(\delta, t) = 4I_0\cos^2\left(\frac{\delta}{2}\right)\cos^2\left(\omega t + \frac{\delta}{2}\right), \tag{3}$$

where I_0 is a combination of ξ_0^2 and whatever constants of proportionality are necessary to convert amplitude to intensity. According to Eq. (3), the wave intensity at a given point fluctuates between its peak value and zero with frequency $\omega/(2\pi)$, unless the intensity is always zero due to the $\cos^2(\delta/2)$ factor. Typically such fluctuations occur too rapidly to be detected, so it is the intensity "time-averaged" over a single period that is of interest. It can be shown that the time-average of $I(\delta t)$ results in the replacement of the $\cos^2(\omega t + \delta/2)$ factor in Eq. (3) with a factor of 1/2.[3] Thus the average intensity, represented by $I(\delta)$, can by expressed as:

$$I(\delta) = 2I_0\cos^2\left(\frac{\delta}{2}\right). \tag{4}$$

This quantity depends only on the phase difference δ, which in turn depends only on the path difference Δ.

3d. Conditions for Maximum and Minimum Intensity. As we have seen, the quantity which determines the amount of interference that occurs when combining wave disturbances from two equal coherent sources is the phase difference δ. The intensity of the net wave disturbance will be zero when $\cos^2(\delta/2)$ is zero. This occurs when δ is 180° or π radians, and when δ is 180° plus any integer multiple of 360°, i.e. when $\delta = 2\pi(n + 1/2)$ radians or $\delta = 360°(n + 1/2)$, where n can take the possible values 0, 1, 2, ...[4] Similarly, the intensity is maximized when $\cos^2(\delta/2) = 1$. A few moments consideration reveal that this occurs when δ is 0°, 360°, or any integer multiple of 360°, i.e. $\delta = 2\pi n$ radians or $\delta = (360°)n$ for $n = 0, 1, 2, \ldots$. Thus to summarize, the intensity of the wave disturbance is maximized or minimized according to the following conditions:

$$\delta = \left\{\begin{array}{ll} 2\pi n, & \text{for maxima} \\ \\ 2\pi\left(n + \dfrac{1}{2}\right), & \text{for minima} \end{array}\right\} n = 0, 1, 2, 3, \ldots \tag{5}$$

Alternatively we can express the conditions in terms of the path difference Δ, using the relationship between δ and Δ given in Eq. (1). Using this

[3]See Appendix: Time-Averaged Intensities.

[4]You should take a moment and convince yourself that these values of δ do indeed produce a zero value for $\cos(\delta/2)$.

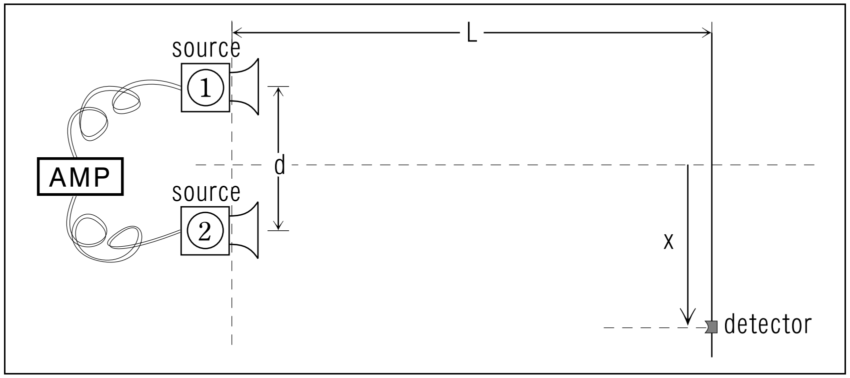


Figure 4. A schematic diagram of a two-source experiment.

relationship, the interference conditions may be written as:

$$\Delta = \left\{ \begin{array}{ll} n\,\lambda, & \text{for maxima} \\ \left(n + \frac{1}{2}\right)\lambda, & \text{for minima} \end{array} \right\} n\,=\,0,\,1,\,2,\,3,\ldots \qquad (6)$$

where the path difference Δ is given by:

$$\Delta = r_2 - r_1$$

where r_1 and r_2 are the distances traveled by waves 1 and 2 respectively.

3e. Calculating the Path Difference. Our problem has now reduced to calculating the path difference for the particular situation we are involved with. This is essentially a problem of geometry and will be different for each situation. Many well known interfering systems have been worked out in detail by physicists and mathematicians over the years. As an example, such a system is discussed in the next section. However you should realize that Eqs. (4)-(6) are the principal results of our derivations and constitute the real physics content of this module.

4. A Two-Source Experiment

4a. Coherent Sources are the Key. An absolutely essential requirement for interference to be observed is the availability of coherent sources of waves. By using two small coherent sources, separated by distance d (measured from the center of each source), an interference pattern can

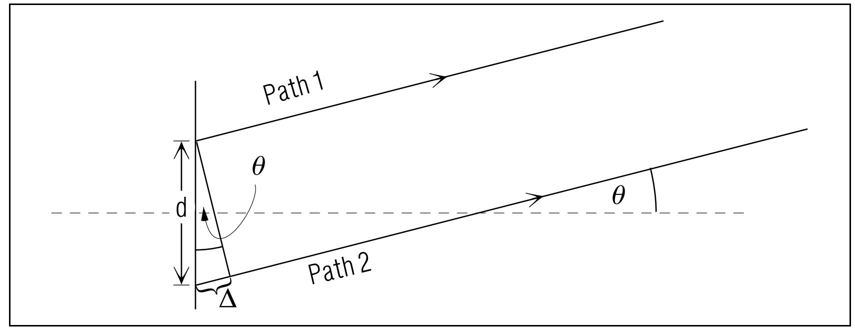


Figure 5. The calculation of the path difference for parallel path distances.

be observed with a receiver a distance L meters away from and moving parallel to the plane of the two sources (see Fig. 4).

4b. An Approximation For the Path Difference. Here is an approximation sometimes used to compute the path difference of any two waves that combine at a receiver. If L is very much larger than either d or λ, the two path distances, r_1 and r_2 are essentially parallel. Thus as can be seen in Fig. 5, the path difference is simply:

$$\Delta = r_2 - r_1 \simeq d \sin\theta \text{ for } L \gg d\,, \qquad (7)$$

where θ is the "angular position" of the point P on the screen where the waves are combining. Combining this geometrical expression for the path difference of two waves with the criteria for interference maxima and minima, we obtain the result:

$$d\sin\theta_n = \left\{ \begin{array}{ll} n\,\lambda, & \text{for maxima} \\ \left(n + \frac{1}{2}\right)\lambda, & \text{for minima} \end{array} \right\} n\,=\,0,\,1,\,2,\ldots$$

where θ_n is the angular position of the n^{th} maximum or minimum. Remember that Eq. (7) is an approximation, good only if $L \gg d$. The exact path difference will depart from the one calculated using Eq. (7) if d becomes comparable to L, resulting in errors in the predicted positions of the maxima and minima.

4c. Positions of Maxima and Minima. For fixed source separation d and wavelength λ, the angular positions of the maxima and minima depends only on the value of n. The value of n determines the "interference

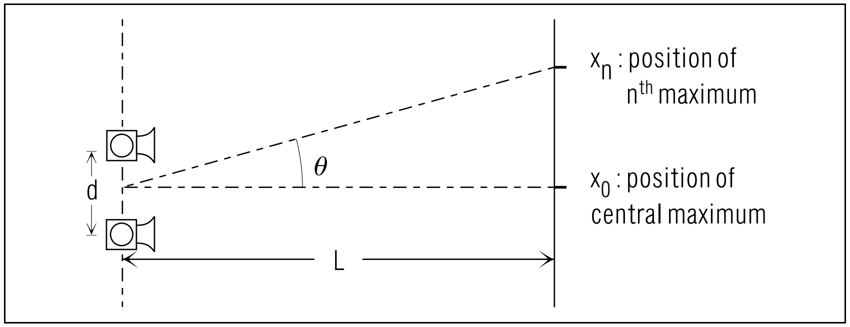


Figure 6. The position of maxima for the two-source experiment.

order" of maxima and minima at the receiver. For $n = 0$, there is only one angle at which there is a maximum; at $\theta = 0$. This maximum is called the "central maximum" and lies on the perpendicular bisector of the line connecting the sources (see Fig. 6). There is a minimum on either side of the central maximum at the angular positions $\theta = \pm \sin^{-1}(\lambda/2d)$. Further maxima and minima appear along the receiver-line at larger angles. These are referred to as the "$n^{\rm th}$ order" maxima and minima, corresponding to the number of complete wavelengths in the path difference. Figure 7 shows a graph of intensity as a function of phase difference, corresponding to a form of Eq. (4) suitably modified to ignore any r^{-2} dependence of the amplitude. The maxima and minima are evenly spaced, characteristic of a cosine-squared function. We could equally well have expressed the intensity as a function of angular or even linear position along the receiver-line, using the relationship between phase difference and path difference and some simple geometry.

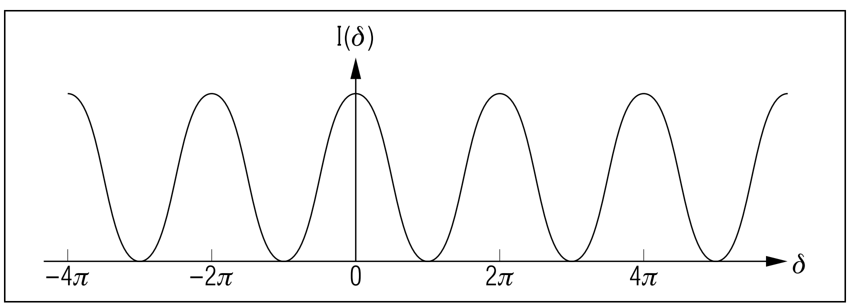


Figure 7. The intensity as a function of phase difference for the two-source experiment.

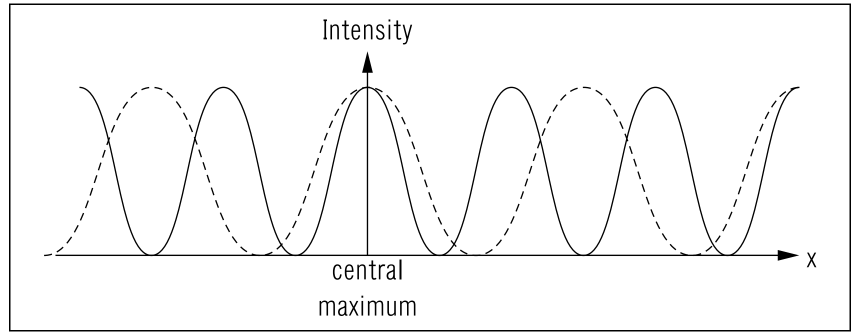


Figure 8. Two overlapping interference patterns.

4d. Overlapping Orders of Interference. If the two sources are producing different wavelengths, we obtain two separate interference patterns superimposed on oneanother. Since the path difference from the source-line to the receiver-line is zero for all wavelengths, the central maxima for both patterns coincide for all wavelengths. However the positions and spacings of other maxima depend on the wavelength, λ, of the waves that are interfering. As the interference order increases, the maxima corresponding to the longer of the two wavelengths are located at greater angular positions from the central maximum than the maxima corresponding to the shorter wavelength. Eventually, the $n^{\rm th}$ maximum of the longer wavelength will overlap the $(n+1)^{\rm th}$ maximum of the shorter wavelength. This is known as "overlapping orders of interference." In Fig. 8 two interference patterns are present in which the $n = 3$ maxima of one pattern occurs at the same position as the $n = 2$ maxima of the other. The difference between a wavelength λ and another, longer wavelength λ' whose $n^{\rm th}$ order maximum coincides with the $(n+1)^{\rm th}$ order maximum of the shorter wavelength is called the "free spectral range" of wavelength λ. This quantity defines the range of wavelengths that can produce interference maxima for an order n without overlapping maxima from other interference orders.

Acknowledgments

I would like to thank Dr. J. S. Kovacs for his contributions to an earlier version of this module. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **central maximum**: the intensity peak in the interference pattern produced by two or more coherent wave sources that corresponds to a combination of waves that have all traveled the same distance, i.e. the path difference is zero.
- **coherent wave sources**: wave sources that produce waves of the same frequency and wavelength, with the same phase at the same time at their respective origins. Also referred to as "synchronized" wave sources.
- **constructive interference**: a combination of two or more wave disturbances at a particular point in space where the net wave intensity is greater than the intensity of any of the individual wave disturbances.
- **destructive interference**: a combination of two or more wave disturbances at a particular point in space where the net wave intensity is less than the intensity of any of the individual wave disturbances.
- **free spectral range**: the range of wavelengths above a given wavelength that can produce interference maxima of a given order that do not overlap maxima of another order of interference.
- **interference pattern**: a spatial variation in the net intensity of the combined wave disturbance due to two or more coherent sources of waves.
- **interference order**: the number of complete wavelengths in the path difference between waves from two successive coherent wave sources that combine at a point in space to form an interference maximum. This number is used to count the maxima, e.g. the third order maxima are the third intensity peaks in the interference pattern on either side of the central maximum.
- **maxima**: points of total constructive interference, where the intensity of the interference pattern is at its greatest. These points occur when the wave disturbances from the various coherent sources present are completely in phase.
- **minima**: points of destructive interference, where the intensity of the interference pattern is at its weakest. These points occur when the phase difference of the individual wave disturbances results in a net disturbance of minimal amplitude.

- **path difference**: the difference in the distance that two waves from successive coherent wave sources travel to reach a given point in space where the wave disturbances combine.
- **phase difference**: the difference in the phase of two wave disturbances that combine at a given point in space. The phase difference is related to the path difference through the wavelength of the wave.
- **phasor**: a fictitious vector that rotates in an imaginary plane and whose component in the x-direction may be used to represent the disturbance produced by a wave at a given point in space. The length of the phasor represents the amplitude of the disturbance and the angle the phasor makes with the x-axis represents the phase of the disturbance.

Time-Averaged Intensities

Only for Those Interested. The intensity of the net wave disturbance produced by two or more coherent wave sources may be written as:

$$I(\vec{r},t) = I(\vec{r})\,\cos^2(\omega t\,+\,\phi)$$

where all spatial coordinate dependencies are included in the time-independent factor $I(\vec{r})$. Since the maximum value of $\cos^2\theta$ is 1 and its minimum value is zero, the intensity at a fixed value of $\vec{r}$ oscillates back and forth between its peak value of $I(\vec{r})$ and its minimum value of zero. The angular frequency of this oscillation is ω (see Fig. 9). These fluctuations are usually too rapid to be easily measured, so measurements are made of the time-averaged intensity.

The long term time-average is the same as the average over one wave period, since all wave periods are duplicates of each other, so we need only calculate the average over one period. The time-average of the intensity is denoted $I_{av}(\vec{r})$ and is defined by:

$$I_{av}(\vec{r}) = \frac{1}{T}\int_0^T I(\vec{r},t)\,dt = \frac{1}{T}\int_0^T I(\vec{r})\,\cos^2(\omega t+\phi)\,dt,$$

where $T = 2\pi/\omega$, which is one period of the wave oscillation. For radio waves, one period is in the region of microseconds (10^{-6} s).

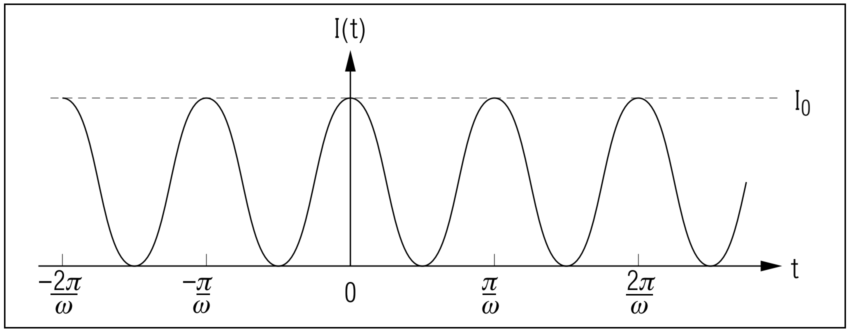


Figure 9. The variation in wave intensity as a function of time at a given point in space at which interference occurs. The time origin was arbitrarily chosen to coincide with an intensity peak.

If we let $\omega t + \phi = x$ so $\omega dt = dx$, and use $\omega T = 2\pi$, the integral assumes this form:

$$I_{av} = \frac{I(\vec{r})}{2\pi}\int_{\phi}^{2\pi+\phi}\cos^2 x\,dx = \frac{I(\vec{r})}{2\pi}\int_0^{2\pi}\cos^2 x\,dx$$

where we have used the fact that the integral over any one period of extent T is the same as the integral over any other period of extent T.

We now use the fact that the integral of $\cos^2$ over a complete period is exactly the same as the integral of $\sin^2$ over a complete period (one looks just like the other, on a graph, but shifted.) Then we can rewrite the integral over $\cos^2$ as half the integral over the sum of the $\sin^2$ plus the $\cos^2$. However, the sum of those two quantities is just 1, so we get:

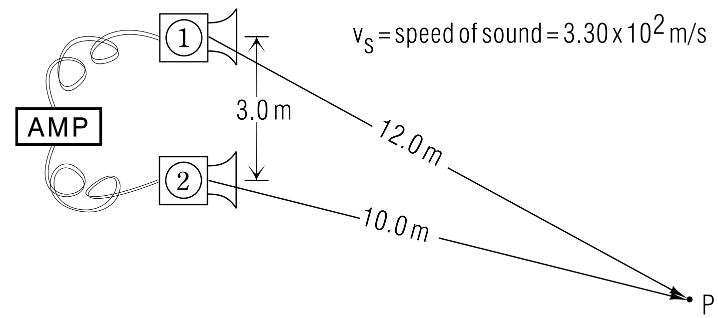
$$I_{av} = \frac{I(\vec{r})}{2\pi}\,\frac{1}{2}\int_0^{2\pi} 1\,dx = \frac{1}{2}\,I(\vec{r}).$$

Thus the time-average of an intensity varying as $cos^2(\omega t + \phi)$ is one-half the peak intensity. Of course the same result holds if the time dependence is $\sin^2(\omega t + \phi)$.

PROBLEM SUPPLEMENT

Problem 8 also occurs in this module's *Model Exam*.

1.



Two identical speakers are connected to the left and right channel output of a stereo amplifier. A single-frequency tone, $\nu = 660\,\text{Hz}$, is played over the system. A point P is 12.0 m from speaker #1 and 10.0 m from speaker #2.

a. Assuming the speakers are properly connected to the amplifier, i.e. "in phase," they constitute two coherent sources of sound waves. Is the interference occurring at point P constructive or destructive?

b. By reversing the connections on one of the speakers, the speakers are 180° "out of phase": when speaker #1 is producing a wave "crest," speaker #2 is producing a wave "trough." Describe the type of interference now taking place at point P.

2. The wave disturbances, at a particular point in space, produced by two wave sources, are given by the equations:

$$\xi_1(r_1, t) = \xi_0\,\sin(kr_1 + \omega t + \phi_1)$$
$$\xi_2(r_2, t) = \xi_0\,\sin(kr_2 + \omega t + \phi_2)$$

The sources are coherent if the frequencies are the same (they are) and if the phase constants are the same, i.e. $\phi_1 = \phi_2$. Suppose that $\phi_1 = \phi_2 = -\pi/2$. This means that, at $t = 0$ and $r_1 = r_2 = 0$:

$$\xi_1(0,0) = \xi_2(0,0) = \xi_0\,\sin(-\pi/2) = -\xi_0$$

which means that a wave "trough" is leaving each source. Sketch ξ_1, ξ_2 and the net wave disturbance, $\xi = \xi_1 + \xi_2$, as functions of ωt when:

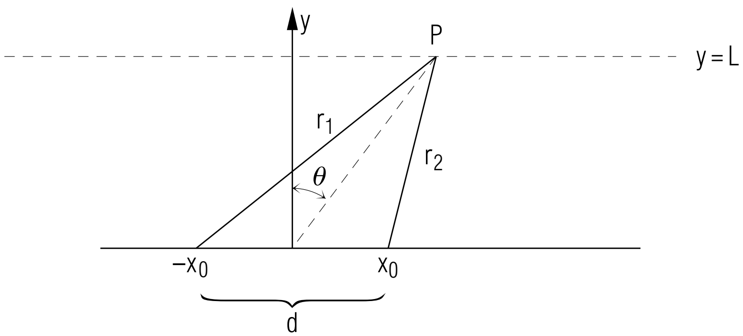
a. $r_1 = \lambda/4,\ r_2 = 3\lambda/4$

b. $r_1 = \lambda/4,\ r_2 = 5\lambda/4$

c. $r_1 = \lambda/4,\ r_2 = \lambda$. Characterize the type of interference for each situation.

3. A point in space is 45.0 meters from one wave source and 45.4 meters from another wave source. The two wave sources are equal and coherent, and produce waves of frequency 165 Hz and wavelength 2.00 meters. Use the phasor method to graphically compute the net wave intensity at the given point in terms of the intensity of either individual wave. Include an accurate phasor diagram constructed with a ruler and a protractor and measure the length of the resultant with the ruler. Compare your graphically obtained answer to the one obtained by using Eq. (1) from the module text, assuming that $r = r_0$.

4.



Two equal coherent wave sources are located at (x_0,0) and ($-x_0$,0), separated by a distance d. Using geometry and the condition for constructive interference, derive a formula for the exact positions, x, of interference maxima along the line $y = L$. Use no approximations.

5. Derive the result of Problem 4 a different way by noting that the constructive interference condition, $r_1 - r_2 = n\lambda$, where $n\lambda$ is a constant for a given n, is the equation of an hyperbola (actually a family of hyperbolas, since $n = 0, 1, 2, \ldots$). The Cartesian coordinate equation of an hyperbola, centered at the origin with foci at x_0 and $-x_0$, is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where a and b are the semi-major and semi-minor axes of the hyperbola. In terms of r_1 and r_2 an hyperbola can be written:

$$r_1 - r_2 = 2a$$

so $a = n\lambda/2$. Finally the relation $x_0 = (a^2 + b^2)^{1/2}$ defines b, given a and x_0. Use the above information to solve for the positions x where the hyperbolas intersect the line $y = L$. These are the locations of interference maxima along $y = L$. Note that the same type of derivation could be carried out to locate interference minima using the relation:

$$r_1 - r_2 = \left(n + \frac{1}{2}\right)\lambda,\ n = 0, 1, 2 \ldots$$

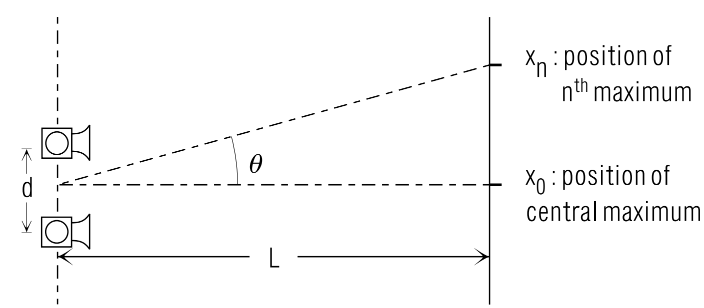
6. Repeat the derivation of Problem 4 assuming that $L \gg d \gg \lambda$, and by dropping negligibly small terms, derive the large-distance result:

$$d \sin\theta = n\,\lambda. \qquad \textit{Help: [S-2]}$$

7. In the two-source interference pattern, maxima and minima are determined by the equations:

$$d \sin\theta_n = \left\{ \begin{array}{ll} n\,\lambda, & \text{for maxima} \\ \left(n + \frac{1}{2}\right)\lambda, & \text{for minima} \end{array} \right\} n = 0, 1, 2, \ldots$$

with the condition that $L \gg d \gg \lambda$ (see diagram below). Thus for reasonably small n, $\sin\theta \ll 1$, so $\sin\theta \approx \tan\theta \approx \theta$, where θ is in radians. Use these relations plus the basic geometrical setup below to determine the spacing between adjacent maxima and between adjacent minima.



8. Consider two equal coherent wave sources with wavelength $\lambda = 5.86\,\text{m}$, located at $x = \pm 4.00\,\text{m}$, $y = 0$.

   a. Determine whether the amount of wave disturbance at the point $x_0 = 8.00\,\text{m}$, $y_0 = 6.93\,\text{m}$ is a minimum or a maximum. Draw a rough sketch of the geometrical layout and show all the steps involved in obtaining the answer.

   b. Sketch a rough graph of the wave disturbance at the point $x_1 = 1.00\,\text{m}$, $y_1 = 3.00\,\text{m}$ as a function of time. Make $t = 0$ be the time at which a wave crest from the source at $x = -4.00\,\text{m}$, $y = 0$ arrives at the point (x_1,y_1). Also show on the graph the disturbance which each source alone would have produced.

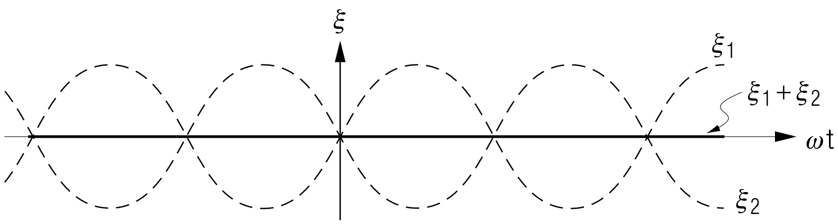
   c. Locate a point of minimum intensity along the line $y_2 = 3.00\,\text{m}$ if the two sources are located as above but have wavelengths of $\lambda = 2.00\,\text{m}$. Show all necessary reasoning.

Brief Answers:

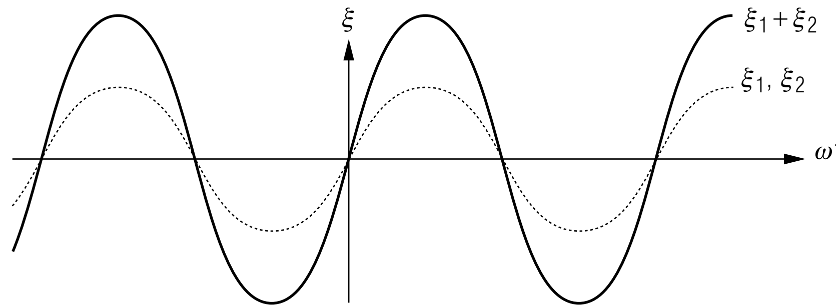
1. a. constructive interference

   b. destructive interference

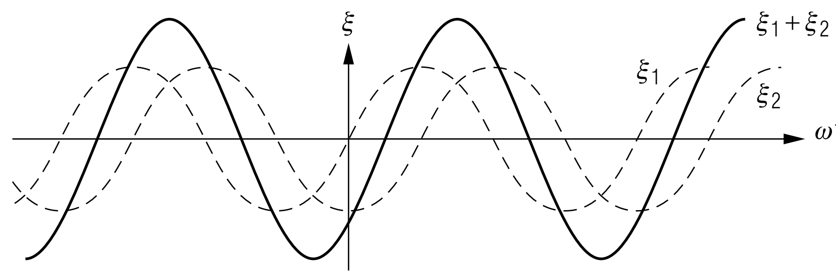
2. a. Total Destructive Interference



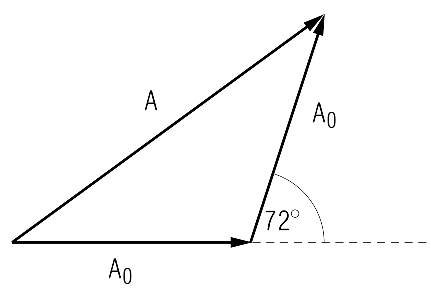
   b. Total Constructive Interference



   c. "Intermediate" Interference *Help: [S-3]*



3. By formal calculation: $I = 1.309\, I_0$



4.

$$x_n = \pm \left(\frac{n\lambda}{2}\right) \left[1 + \frac{L^2}{\left(\frac{d}{2}\right)^2 - \left(\frac{n\lambda}{2}\right)^2} \right]^{1/2}$$

Help: [S-1]

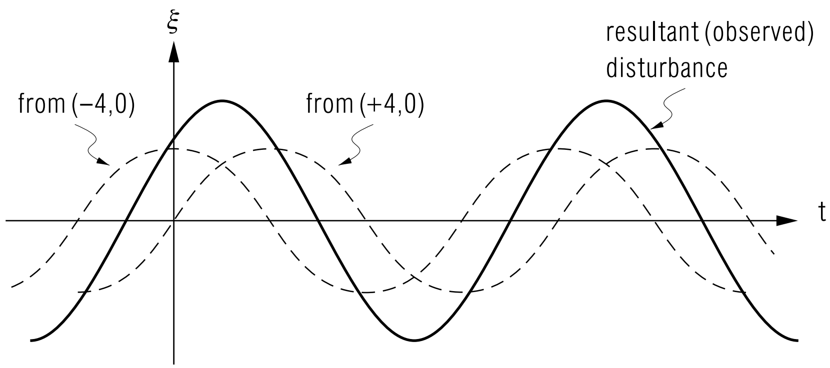
5. Same as Answer 4.

6. $d \sin\theta = n\lambda$.

7. $\Delta x = \lambda L/d$ for adjacent maxima; $\Delta x = \lambda L/d$ for adjacent minima.

8. a. Maximum

   b. The wave from (-4.00 m,0) will be slightly behind the wave from (+4.00 m,0) hence:



   c. $x = \pm 0.63$ m are the closest minima to the y-axis. Of course there are other minima further out. If you get an answer other than the one given, substitute your answer into your original expressions for the two paths. Then either: (i) their difference is not the half wavelength it should be so your algebra was wrong; or (ii) your original expressions were wrong.

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS, problem 4)*

$r_1 - r_2 = n\lambda$, where r_1 and r_2 are defined in terms of somewhat unwieldy square roots. Be sure to move r_2 to the other side of the equation before squaring to solve for x, i.e.

$$r_1^2 = (r_2 + n\lambda)^2 = r_2^2 + 2n\lambda r_2 + (n\lambda)^2.$$

Substituting the expressions for r_1^2 and r_2^2 (leaving r_2 as is, temporarily) several terms cancel. You can then isolate r_2 on one side and square again, then solve for x.

S-2 *(from PS-problem 6)*

Somewhere in your derivation you should get a term similar to:

$$x^2 \left[\frac{d^2}{n^2\lambda^2} - 1 \right] = L^2 + \left(\frac{d}{2}\right)^2 - \left(\frac{n\lambda}{2}\right)^2 .$$

For $L \gg d \gg \lambda$, the 2nd and 3rd terms of the right side of this equation are negligible, particularly after squaring each term. To convince your self, substitute some typical numbers: $L = 10\,\text{m}$, $d = 0.1\,\text{mm}$, $\lambda = 500\,\text{nm}$ for $n = 1$ (any reasonable low value of n will do). Therefore dropping these two terms and noting that $x = L\tan\theta$ exactly, and that

$$1 + \tan^2\theta = sec^2\theta,$$

where $\sec\theta = 1/\cos\theta$, the derivation can easily be finished.

S-3 *(from PS-problem 2c)*

The curves plotted represent:

$$\begin{aligned} \xi_1 &= \xi_0 \sin(\omega t) \\ \xi_1 &= \xi_0 \sin(\omega t + 3\pi/2) \\ &= \xi_0 \sin(\omega t - \pi/2). \end{aligned}$$

Note that one must use: $k = 2\pi/\lambda$.

MODEL EXAM

1. See Output Skills K1-K3 in this module's *ID Sheet*. One or more of these skills, or none, may be on the actual exam.

2. Consider two equal coherent wave sources with wavelength $\lambda = 5.86\,\mathrm{m}$, located at $x = \pm 4.00\,\mathrm{m}$, $y = 0$.

   a. Determine whether the amount of wave disturbance at the point $x_0 = 8.00\,\mathrm{m}$, $y_0 = 6.93\,\mathrm{m}$ is a minimum or a maximum. Draw a rough sketch of the geometrical layout and show all the steps involved in obtaining the answer.

   b. Sketch a rough graph of the wave disturbance at the point $x_1 = 1.00\,\mathrm{m}$, $y_1 = 3.00\,\mathrm{m}$ as a function of time. Make $t = 0$ be the time at which a wave crest from the source at $x = -4.00\,\mathrm{m}$, $y = 0$ arrives at the point (x_1, y_1). Also show on the graph the disturbance which each source alone would have produced.

   c. Locate a point of minimum intensity along the line $y_2 = 3.00\,\mathrm{m}$ if the two sources are located as above but have wavelengths of $\lambda = 2.00\,\mathrm{m}$. Show all necessary reasoning.

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 8.