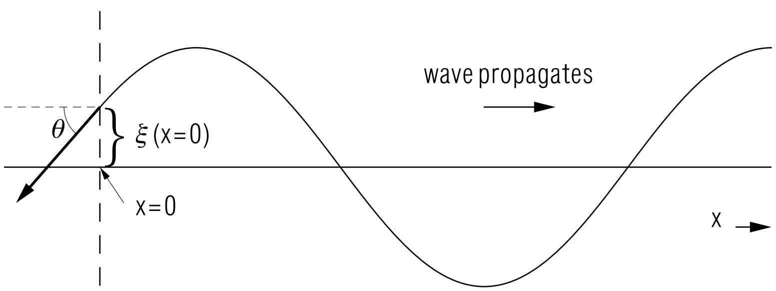
PHYSNET

MISN-0-203

INTENSITY AND ENERGY IN SOUND WAVES



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

INTENSITY AND ENERGY IN SOUND WAVES

by

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Title: **Intensity and Energy in Sound Waves**

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Input Skills:

1. Vocabulary: compressions, rarefactions, longitudinal wave, transverse wave, sound wave (MISN-0-202); energy, power (MISN-0-20).
2. For transverse waves in a stretched string and longitudinal compressional waves in a solid, relate the wave speed and the force on the individual particles of the medium to the physical properties of the material transmitting the wave (MISN-0-202).

Output Skills (Knowledge):

K1. Vocabulary: decibel, plane wave, spherical wave, wave intensity, wave vector, wave front.

K2. State the expression for the intensity of a plane wave and the power transmitted across a given cross-sectional area of the medium through which the wave travels.

Output Skills (Rule Application):

R1. For intensity calculations associated with plane and spherical waves, use the unit of intensity level, the decibel.

Output Skills (Problem Solving):

S1. For an acoustic plane wave of given amplitude and frequency, traveling through an elastic medium of given mass density and elasticity, calculate the wave's intensity, energy density, and rate at which it propagates energy.

S2. Given the intensity of a spherical wave at one radial distance from the wave source, calculate the wave's intensity at any other radial distance.

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1. Introduction

In a previous module we have seen that sound (acoustic) waves are the result of periodic disturbances of an elastic medium.[1] These disturbances may be represented by variations in the local pressure of a region in the medium (compressions and rarefactions) or by the displacement of portions of the medium from their equilibrium position. However, this displacement is in the form of a simple harmonic oscillation, whether it be a transverse displacement as with waves on a stretched string or a longitudinal displacement as with sound waves in a gas. Either way there is no net movement of matter. It would seem appropriate then to ask what it is that is "traveling" when a traveling wave passes through a medium.

2. One-Dimensional Elastic Waves

2a. Energy is What is Propagated. When a one-dimensional traveling wave propagates through an elastic medium, the wave carries energy in the direction that the wave travels. This energy is the kinetic and potential energy of the deformation of the elastic medium. The total mechanical energy of an infinitesimal mass element dm in the elastic medium is

$$dE = \frac{1}{2}\omega^2 \xi_0^2 \, dm \,, \tag{1}$$

where ω is the angular frequency of the wave and ξ_0 is the displacement amplitude of the wave.[2] Depending on the geometry of the medium this energy may be expressed as one of several energy densities. For a stretched wire of linear mass density μ, the mass element dm may be expressed as:

$$dm = \mu \, dx \,. \tag{2}$$

[1]See "Sound Waves and Small Transverse Waves on a String" (MISN-0-202).
[2]For a detailed derivation of this relation, see the Appendix.

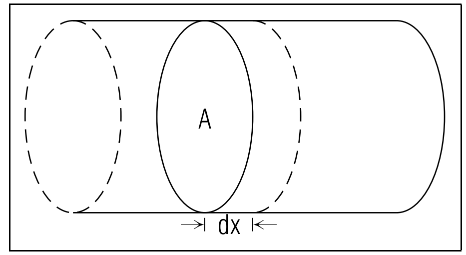

Figure 1. An infinitesimal volume element of an elastic medium of constant cross-sectional area.

Substituting this expression for dm into Eq. (1) and dividing by dx,[3] we may define a linear energy density E_ℓ:

$$E_\ell = \frac{dE}{dx} = \frac{1}{2}\mu\omega^2\xi_0^2 \,. \tag{3}$$

Similarly, for energy propagating through a three-dimensional elastic medium of constant cross-sectional area, such as a solid rod or a column of gas, a volume energy density E_v may be defined as

$$E_v = \frac{dE}{dV} = \frac{1}{2}\rho\omega^2\xi_0^2 \,, \tag{4}$$

where ρ is the volume density of the elastic medium.

2b. Power is Required to Maintain a Train of Waves. For a continuous series of sinusoidal pulses, called a "wave train," to be maintained in an elastic medium, the energy that propagates in the medium must be supplied by some external agent. The rate at which this energy is supplied is the "power" of the wave source, defined as

$$P = \frac{dE}{dt} \,. \tag{5}$$

For a transverse wavetrain propagating on a stretched wire, we may use the chain-rule of differential calculus to express Eq. (5) as

$$P = \frac{dE}{dt} = \frac{dE}{dx}\frac{dx}{dt} = E_\ell v \,, \tag{6}$$

or

$$P = \frac{1}{2}\mu\omega^2\xi_0^2 v \,, \tag{7}$$

where $v = (T/\mu)^{1/2}$ is the speed with which energy propagates along the wire, the wave speed. A similar expression may be derived for a wave

[3]See "Sound Waves and Small Transverse Waves on a String" (MISN-0-202).

traveling through a three-dimensional medium of constant cross-sectional area. If an infinitesimal volume element dV is chosen with cross-sectional area A and thickness dx, where the x-direction is chosen parallel to the direction of wave propagation, then $dV = A\,dx$. Thus

$$P = \frac{dE}{dt} = \frac{dE}{dx}\frac{dx}{dt} = \frac{dE}{dV}\frac{dx}{dt}A\,,$$

so:

$$P = E_v\,A\,v\,, \tag{8}$$

or

$$P = \frac{1}{2}\rho\omega^2\xi_0^2 Av\,, \tag{9}$$

where $v = (Y/\rho)^{1/2}$ for longitudinal waves on a solid rod or $v = (K/\rho_0)^{1/2}$ for longitudinal waves in a column of gas.

2c. Definition of Wave Intensity. For cases of a one-dimensional wave traveling in a three-dimensional medium of constant cross-sectional area, an important quantity called the "wave intensity" may be defined as the power per unit cross-sectional area, I:

$$I = \frac{P}{A}\,. \tag{10}$$

Substituting Eq. (9) into Eq. (10), the intensity of a one-dimensional wave propagating in an elastic medium of constant cross-sectional area may be written as:

$$I = \frac{1}{2}\rho\omega^2\xi_0^2 v\,, \tag{11}$$

which is constant for waves of given amplitude and frequency. Intensity is the physical quantity which, for a sound wave, roughly corresponds to the "loudness" or "softness" of the sound. Since the MKS unit power is the watt (W), the MKS unit of intensity is the watt per square meter (W/m 2).

2d. The Intensity Level of Sound. Because the range of intensities of audible sounds is so wide, sound intensities are often expressed using a logarithmic scale, referred to as the "intensity level." Intensity levels are determined using the following equation:

$$I(\,\text{db}) = 10\log\left(\frac{I}{I_{\text{ref}}}\right)\,, \tag{12}$$

where I is the intensity of interest, I_{ref} is a reference intensity, and $I(\,\text{db})$ is the intensity level of the intensity I in units of "decibels," abbreviated

"db." For sound waves in air, I_{ref} has arbitrarily been chosen to be $10^{-12}\,\text{W/m}^2$. A list of typical sounds and their corresponding intensities and intensity levels are shown in Table 1.

Table 1. The Intensities of Some Sounds.

SOUND	INTENSITY (W/m^2)	RELATIVE INTENSITY (db)
Threshold of hearing	10^{-12}	0
Rustling leaves	10^{-10}	20
Talking (at 3 ft.)	10^{-8}	40
Noisy Office or Store	10^{-6}	60
Elevated train	10^{-4}	80
Subway car	10^{-2}	100
Threshold of pain	1	120

3. Plane Waves

3a. Definition of a Plane Wave. A one-dimensional wave for which the wave disturbance is distributed uniformly over a planar surface (either finite or infinite), is called a "plane wave." If the surface is of finite extent, the plane wave is said to be "collimated." The wave is still one-dimensional as long as it travels in a single direction. Figure 2a is an illustration of how we visualize a plane wave as a series of parallel plane surfaces, called "wave fronts" moving in the direction indicated. A wave

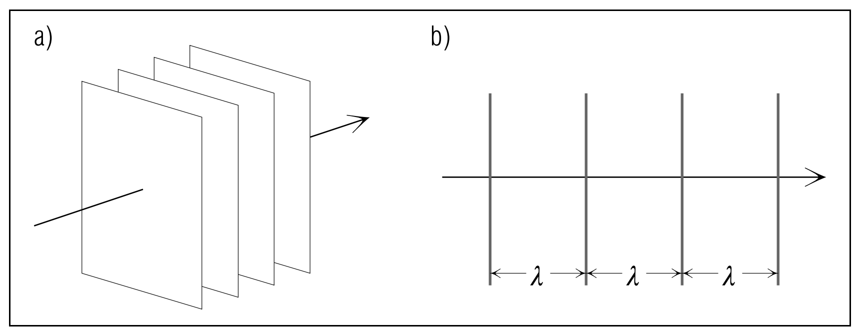

Figure 2. An illustration of a plane wave: a) oblique view; b) cross-sectional view.

front is a surface over which the wave disturbance has the same phase, i.e. for a plane wave traveling in the x-direction:

$$\phi = kx - \omega t + \phi_0 = \text{ constant} \,. \tag{13}$$

Figure 2b shows a cross-sectional view of the wave fronts of a plane wave. We typically draw such a sketch with the wavefronts one wavelength apart, so that they are all crests, or all troughs, or any other given wave disturbance.

3b. The Wave Vector. In order to describe a plane wave propagating in any direction, we will define a quantity called the "wave vector." The wave vector, symbolized by $\vec{k}$, is a quantity whose magnitude is the wave number, k, of the wave, and whose direction is the direction of propagation of the wave.[4] Using the wave vector we may define the phase of a plane wave as:

$$\phi = \vec{k} \cdot \vec{r} - \omega t + \phi_0 \,, \tag{14}$$

where $\vec{r}$ is the position vector of a point on a particular wave front with respect to a specified coordinate system. Thus the wave function for a plane wave may be written as:

$$\xi = \xi_0 \sin(\vec{k} \cdot \vec{r} - \omega t + \phi_0) \,. \tag{15}$$

3c. The Intensity of a Plane Wave. The intensity of a plane wave is defined as the power propagated per unit area of wave front. Since the power is distributed uniformly over the wave front, the intensity is constant for plane waves of given amplitude, frequency, and wave speed. Thus one-dimensional waves travelling in an elastic medium of constant cross-sectional area are examples of collimated plane waves, and the relations derived for the energy density and the intensity of these waves are applicable to plane waves in general. Note that if the wave front of the plane wave is of infinite extent, the total power propagating across its surface is also infinite, although the energy density and intensity are not. However, for systems of physical interest, only a finite portion of the wave front impinges on a system capable of detecting the power propagated, so the power detected is finite. Furthermore, the concept of a plane wave front of infinite area is usually an idealized approximation.

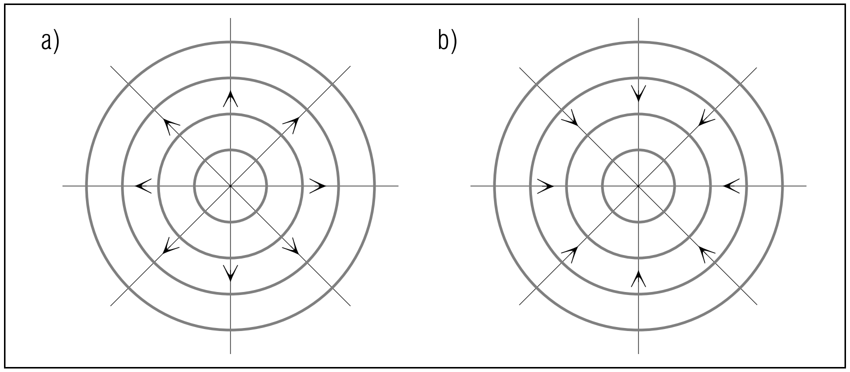

Figure 3. (a) A spherical wave propagating outward; (b) a spherical wave propagating inward.

4. Spherical Waves

4a. Wave Front of a Spherical Wave. Another type of wave frequently encountered is the "spherical wave." In contrast to a plane wave, the wave fronts of a spherical wave are concentric spherical surfaces. The surfaces travel radially outward or inward, depending on the sign of the frequency term in the phase of the wave function (see Fig. 3).

4b. Intensity of a Spherical Wave. The intensity of a spherical wave is defined as the power propagated per unit area of wave front, just as for a plane wave. However, the power and energy are distributed over a spherical wave front of area $4\pi r^2$. Since each wave front is expanding radially outward (or contracting inward) as the wave propagates, the intensity varies as r^{-2} for a spherical wave. Assuming the power output of the wave source, P_0, is a constant, the intensity of the spherical wave is given by

$$I = \frac{P_0}{4\pi r^2} \,. \tag{16}$$

If the intensity is known at a specific radial distance r_0, then since $P_0 = I_0 4\pi r_0^2$, the intensity at any other radial distance r may be expressed as:

$$I = I_0 \left(\frac{r_0}{r}\right)^2 \,. \tag{17}$$

4c. The Wave Function of a Spherical Wave. By solving the wave equation for a wave source of spherical symmetry, the wave function for

[4] Note: $k = \omega/v$.

a spherical wave can be shown to be:

$$\xi(r,t) = \xi_0 \left(\frac{r_0}{r}\right) \sin(kr - \omega t + \phi_0)\,, \tag{18}$$

where ξ_0 and ϕ_0 are determined from the boundary conditions.[5] If we consider the entire coefficient of the sine function to be the "amplitude," then we see that the amplitude of a spherical wave decays as r^{-1}. Since the wave intensity is proportional to the square of the wave amplitude, for a spherical wave the intensity is:

$$I = \beta\xi_0^2 \left(\frac{r_0}{r}\right)^2 . \tag{19}$$

This expression is equivalent to Eq. (17) if $I_0 = \beta\xi_0^2$. The constant of proportionality, β, depends on the specific medium through which the wave travels and the physical nature of ξ_0.

Acknowledgments

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Glossary

- **decibel**: a unit of intensity on a logarithmic scale of "intensity level," abbreviated as "db." An additive increase in the intensity level of 10 db implies a multiplicative increase in the actual intensity by a factor of 10.
- **plane wave**: a one-dimensional wave traveling in a direction defined by the wave vector $\vec{k}$, whose surfaces of equal phase are parallel planes of finite or infinite extent.
- **spherical wave**: a three-dimensional wave emanating from a wave source of spherical symmetry, whose surfaces of equal phase are concentric spheres.

[5] Note that the actual value of ϕ_0 will depend on whether a sine or a cosine function is used.

- **wave intensity**: the power propagated by the wave per unit area perpendicular to the propagation direction; the units of intensity are $\mathrm{W/m^2}$.
- **wave front**: a continuous surface of wave disturbances of the same phase, such as a crest or a trough.
- **wave vector**: a vector whose magnitude is the wave number k and whose direction is the direction of wave propagation.

Energy Density of a 1D Elastic Wave

Only For Those Interested. To derive the energy density of a one-dimensional elastic wave traveling in an elastic medium of constant cross-sectional area, consider an infinitesimal volume element of the medium. The total energy in this element is:

$$dE = dE_K + dE_P \tag{20}$$

where dE_K is the kinetic energy due to the motion of the infinitesimal volume element and dE_P is the potential energy of the element's displacement from equilibrium. If the infinitesimal volume element has mass dm, the kinetic energy may be represented as

$$dE_K = \frac{1}{2}\dot{\xi}^2\, dm \tag{21}$$

where $\dot{\xi} = \partial\xi/\partial t$, the speed of the medium's deformation. For a one-dimensional sinusoidal wave, ξ may be represented as:

$$\xi(x,t) = \xi_0 \sin(kx - \omega t + \phi_0) \tag{22}$$

so

$$\dot{\xi} = \pm\omega\xi_0 \cos(kx - \omega t + \phi_0). \tag{23}$$

The potential energy of the mass element may be represented as:

$$dE_P = \frac{1}{2}\xi^2\, dk \tag{24}$$

where dk is the elastic constant of the restoring force acting on the infinitesimal mass element. This force constant may be identified by applying Newton's second law to the mass element. The force on the mass element is given by:

$$dF = dm\frac{\partial^2\xi}{\partial t^2} = -\omega^2 dm\xi_0 \sin(kx - \omega t + \phi_0) = -\omega^2 dm\xi. \tag{25}$$

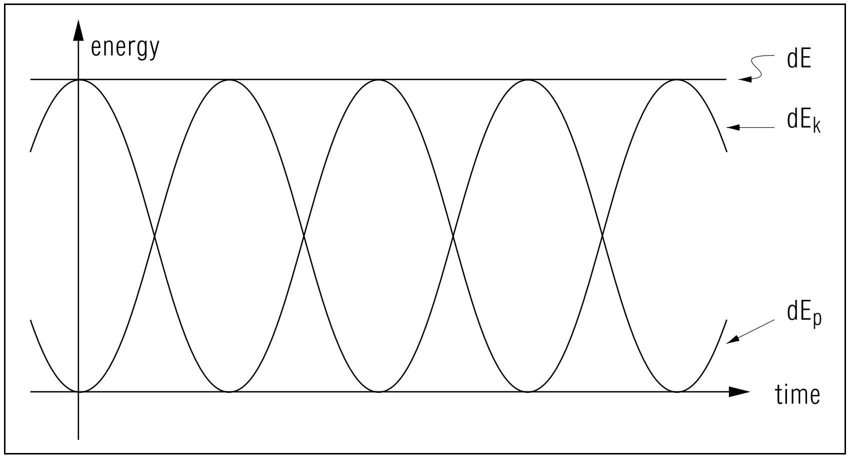


Figure 4. The energy of a mass element dm in an elastic medium, as a function of time, as a one-dimensional wave passes by.

This result is a form of Hooke's law, so the force constant must be

$$dk = \omega^2 dm. \qquad (26)$$

Therefore the potential energy of the mass element is:

$$dE_P = \frac{1}{2}\xi^2\omega^2\, dm = \frac{1}{2}dm\omega^2\xi_0^2 \sin^2(kx - \omega t + \phi_0), \qquad (27)$$

and the kinetic energy of the mass element is:

$$dE_K = \frac{1}{2}dm\omega^2\xi_0^2 \cos^2(kx - \omega t + \phi_0). \qquad (28)$$

Since $\sin^2\theta + \cos^2\theta = 1$, the total energy of the mass element is:

$$dE = \frac{1}{2}\omega^2\xi_0^2 dm. \qquad (29)$$

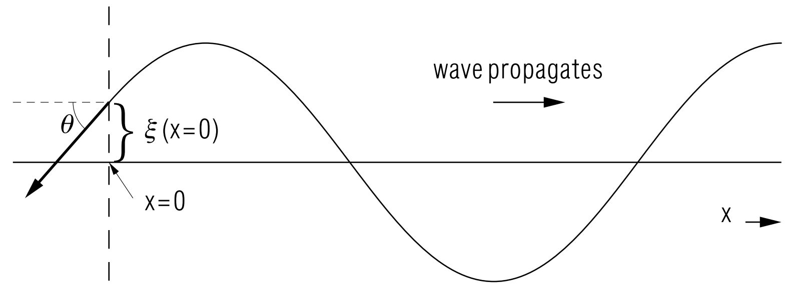
The results of Eqs. (27) - (29) are illustrated in Fig. 4. You can see that at a given point in the medium the energy oscillates between kinetic and potential energy; yet the total energy remains constant, dependent only on the mass of the infinitesimal element, the angular frequency of the harmonic oscillation, and the amplitude of the wave displacement.

PROBLEM SUPPLEMENT

$$Y(\text{iron}) = 2.06 \times 10^{11}\ \text{N/m}^2$$
$$\rho(\text{iron}) = 7.86 \times 10^3\ \text{kg/m}^3$$

Problems 4 and 5 also occur on this module's *Model Exam*.

1.



Consider a string of length L, mass M (mass per unit length $\mu = M/L$) under tension T. Suppose the left end of the string is being moved up and down (transversely) such that a sinusoidal wave, of angular frequency ω and wave number k, travels away from this end along the string.

a. What is the component of the applied force in the transverse direction at any particular time, written in terms of the tension T in the string and the angle θ the string makes with the string axis at that time? The energy of motion of the particles on the string is all transverse so that's the direction of the component of the force which provides this energy.

b. For small displacements ξ the angle θ will always be small such that $\sin\theta \approx \tan\theta \approx \theta$ is a good approximation. Therefore, express your answer to (a) in terms of the slope of the curve traced by the string at the point where the driving force acts.

c. Write down the functional form of $\xi(x,t)$ for a sinusoidal wave propagating to the right.

d. Substitute this into your answer to (b). This tells you the time dependence of the transverse force required to generate a traveling sine wave to the right along the string.

e. If this force displaces the point at the end of the string (at $x = 0$) by an amount $d\xi$, how much work is done by the agent exerting this force?

f. Determine the amount of work done by this driving force during one complete period of motion.

g. Find the average rate at which this work is done using the fact that, for transverse waves on a stretched string, $v = \sqrt{T/\mu}$. Express this rate in terms of the mass density of the string.

2. Longitudinal sound waves propagate down a length of iron rod of cross-sectional area $30\,\mathrm{cm}^2$. The average sound energy density distributed throughout the volume of this iron rod is $4\,\mathrm{J/m}^3$.

a. How is this energy density distributed between kinetic energy of oscillation (of the atoms of the rod) and potential energy?

b. What is the speed of sound along this rod?

c. What is the wave intensity along the rod?

d. What power must be supplied to one end of this rod to maintain this energy density in the rod?

3. The intensity of a spherical sound wave emanating from a point source is observed to be $4 \times 10^{-8}\,\mathrm{W/m}^2$ at a distance of one meter from the source.

a. Find the intensity in decibels of the sound that reaches a point 10 meters from the source.

b. Find the total power supplied by the source.

c. What is the total energy per second crossing the sphere of radius 1 meter with the source at the center?

d. Repeat (c) for the sphere of 10 meters.

e. What is the intensity in decibels 100 meters from the source?

4. The differential equation satisfied by transverse waves along a stretched string of mass M, length L, under tension T is, when the string is aligned parallel to the x-direction:

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} = \frac{TL}{M}\frac{\partial^2 \xi(x,t)}{\partial x^2}\,.$$

The string's length is 10 meters, it has a mass of 5 grams, and it is under a tension of 30 newtons. A sinusoidal source at one end of the string sends sinusoidal waves of wavelength 2 meters down the length of the string. The power supplied by the driving oscillator is 3 watts.

a. What is the speed with which waves propagate along this string?

b. What is the amplitude of the transverse waves propagating along the string?

5. A point source of sound emits 50,000 joules of sound energy every 20 seconds. At a distance 100 meters from the source, what is the intensity of the sound (in decibels), if no energy is lost in the intervening space?

Brief Answers:

1. a. $F_y = -T\sin\theta$ *Help: [S-3]*
   b. $F_y = -T(\partial\xi/\partial x)$ at $x = 0$
   c. $\xi(x,t) = \xi_0 \sin(kx - \omega t + \phi_0)$
   d. $F_y = -kT\xi_0\cos(\omega t - \phi_0)$
   e. $dW = F_y d\xi$
   f. $W = \omega T \xi_0^2 \pi/v$ *Help: [S-1]*
   g. $P = \omega^2 T\xi_0^2/(2v)$ *Help: [S-2]*
2. a. Evenly distributed between kinetic and potential.
   b. $v = 5119\,\mathrm{m/s}$
   c. $I = 2.05 \times 10^4\,\mathrm{W/m}^2$
   d. 61.4 watts
3. a. 26 db
   b. 5.03×10^{-7} watts
   c. 5.03×10^{-7} watts
   d. 5.03×10^{-7} watts
   e. 6 db
4. a. $v = 245\,\mathrm{m/s}$.
   b. $\xi_0 = 9$ millimeters.
5. 103 db.

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-problem 1f)*

$$\begin{aligned} W_{\text{one cycle}} &= \oint F_y \, d\xi \\ &= kT\xi_0 \oint \cos\omega t d(\xi_0 \sin\omega t) \\ &= k\omega T\xi_0^2 \int_0^{2\pi/\omega} \cos^2(\omega t)\, dt \\ &= kT\xi_0^2 \int_0^{2\pi} \cos^2(\omega t)\, d(\omega t) \\ &= kT\xi_0^2 \pi \\ &= \frac{\omega}{v} T\xi_0^2 \pi \end{aligned}$$

S-2 *(from PS-problem 1g)*

$\Delta t_{\text{one cycle}} = 2\pi/\omega$

S-3 *(from PS-problem 1a)*

See Module 202, Sect. 3b.

MODEL EXAM

$$I_{ref} = 10^{-12}\,\mathrm{W}/m^2$$

1. See Output Skills K1-K2 in this module's *ID Sheet*. One or more of these skills may be on the actual exam.

2. The differential equation satisfied by transverse waves along a stretched string of mass M, length L, under tension T is, when the string is aligned parallel to the x-direction:

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Brief Answers:

1. See this module's *text*.
2. See Problem 4 in this module's *Problem Supplement*.
3. See Problem 5 in this module's *Problem Supplement*.