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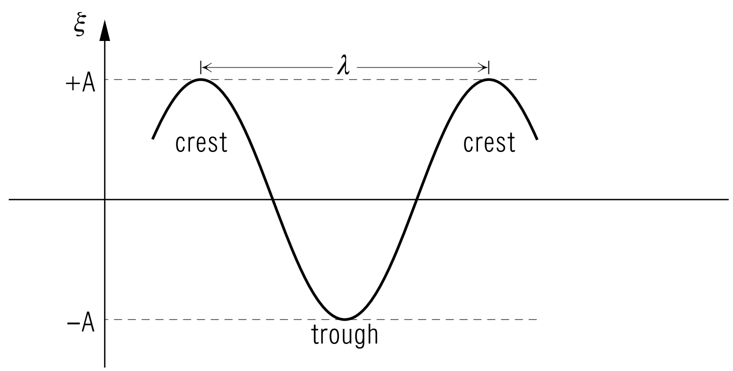
MISN-0-201

THE WAVE EQUATION AND ITS SOLUTIONS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

THE WAVE EQUATION AND ITS SOLUTIONS

by

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Title: **The Wave Equation and Its Solutions**

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Input Skills:

1. Vocabulary: displacement, velocity (MISN-0-7); frequency, angular frequency, period (MISN-0-25).
2. Take the derivative of $\sin(kx+\omega t)$ with respect to t while treating x as a constant whose derivative is zero; alternatively with respect to x while treating t as a constant ((MISN-0-1).

Output Skills (Knowledge):

K1. Vocabulary: amplitude, wavelength, wave number, phase, phase constant, wave function, wave speed, wave equation, harmonic function, sinusoidal wave, traveling wave, boundary conditions, field.

K2. State the one-dimensional wave equation and its general solution.

K3. Take the partial derivative of $\sin(kx + \omega t)$ with respect to either x or t.

Output Skills (Rule Application):

R1. Given a wave function for a one-dimensional traveling wave, verify that it satisfies the wave equation.

Output Skills (Problem Solving):

S1. Given a sufficient number of parameters associated with a sinusoidal wave, write down the mathematical description of the traveling wave.

S2. Determine the unknown parameters of a one-dimensional sinusoidal wave, given its displacement as a function of either: (i) position at two different times; or (ii) time at two different positions.

S3. Determine the unknown parameters of a one-dimensional sinusoidal wave, given the wave function and its first derivative with respect to time at $x = 0$ and $t = 0$.

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1. Overview

Waves and vibrations in mechanical systems constitute one of the most important areas of study in all of physics. Evidence of the existence of these phenomena can be observed for almost any kind of physical system. The propagation of sound and light, ocean waves, earthquakes, the transmission of signals from the brain are a few examples. In this unit we introduce the descriptors of waves and their motions: periodicity, amplitude, propagation speed, etc. We relate these to symbols in the differential form of the wave equation and in its formal solutions. We also relate these descriptors to the properties of some simple physical systems.

2. The Wave Function

2a. Graphical and Mathematical Representation. Waves are represented mathematically by a wave function that may be expressed as a graph or a formal function. This function describes the disturbance made by the wave at various times as it propagates in space. Because the wave function depends on both position and time, when we wish to draw a graph of the wave we usually keep one variable fixed. For example, Fig. 1

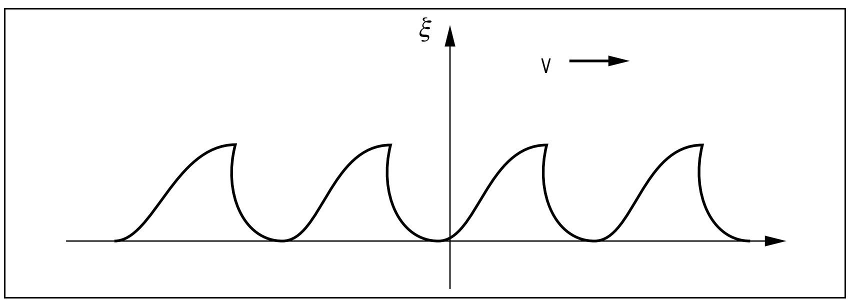


Figure 1. A "snapshot" of a water wave showing the wave profile at a given instant of time. The vertical axis indicates the displacement of water from its average level.

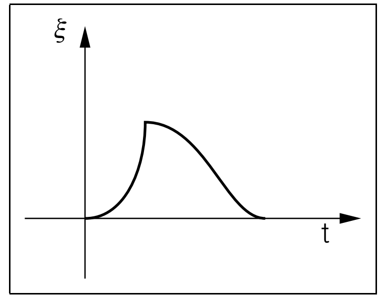


Figure 2. Vertical displacement of a floating object as a water wave passes.

shows the profile of a series of water waves at a given instant of time. You can think of this as a "snapshot" of the wave. Alternatively we can examine the time variation of the wave at a specific point x. If we look at a piece of driftwood as the waves pass, it will bob up and down, executing a periodic motion. The vertical displacement of the driftwood may also be represented graphically, as illustrated in Fig. 2. The exact mathematical form of the wave function, $\xi = f(x,t)$, depends on the type of wave that is being considered.

2b. Traveling Waves. One particular form of wave function, $\xi = f(x - vt)$, corresponds to the "traveling wave," the most common type of wave we encounter. A traveling wave consists of a periodic series of oscillations of some quantity that travel, or "propagate," through space with a speed characteristic of the wave and the medium through which it travels. The expression $\xi = f(x - vt)$ represents a one-dimensional traveling wave propagating in the positive x-direction with speed v. If the oscillations in the medium are simple harmonic oscillations, the functional form of the wave function is also harmonic. It is important to convince yourself that when $f(x)$ is a function representing some curve then the same function f, but with (x) replaced by $(x - a)$, represents a curve with the same shape but shifted along the positive x-axis by an amount "a" (see Fig. 3). Similarly $f(x + a)$ represents that curve shifted in the negative x-direction by an amount "a."

Try it with some simple, easy to compute function.[1] Understanding this is fundamental to understanding the mathematical description of wave motion. Replacing "a" by something linear in time, $a = vt$, gives you a curve that propagates either to the right or the left depending on the sign of v.

[1] For example, $f = ax$ or $f = C \sin bx$.

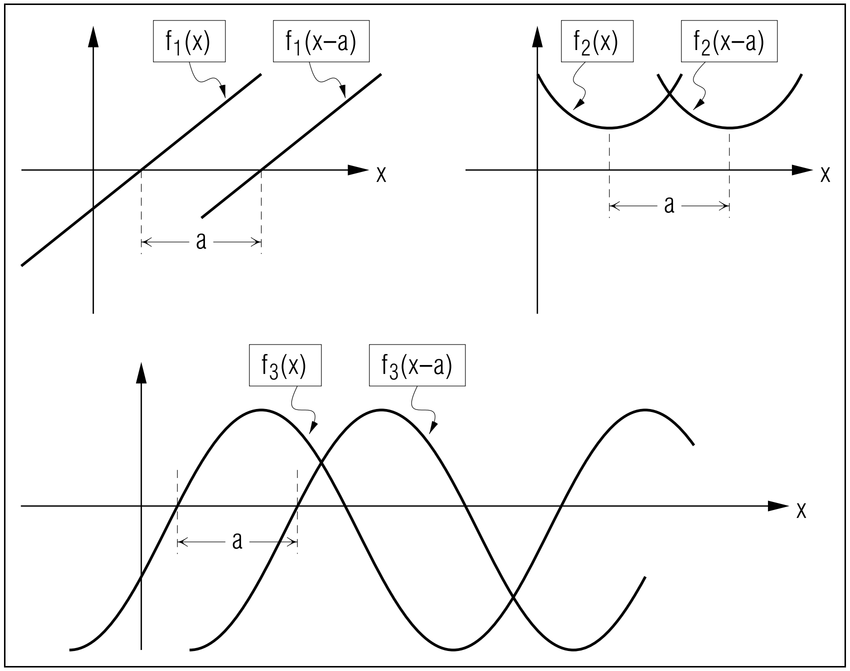


Figure 3. Three different functions shifted to the right by an amount "a": the accompanying replacement of (x) by $(x-a)$ is a general property of all functions.

3. Description of the Wave Motion

3a. The Harmonic Wave Function. A harmonic wave function is sinusoidal in functional form and for a one-dimensional wave may be expressed as either:

$$\xi = A \sin\left[\frac{2\pi}{\lambda}(x \pm vt) + \phi_0\right], \tag{1}$$

or

$$\xi = A \cos\left[\frac{2\pi}{\lambda}(x \pm vt) + \phi_0\right]. \tag{2}$$

This means that the profile of the wave at a particular time is a sine or cosine function, as shown in Fig. 4, and that at a particular point in space, the wave produces a simple harmonic oscillation in some quantity ξ, as indicated in Fig. 5. The symbols in Eqs. (1) and (2) are explained in the remaining paragraphs of this section. The parameters A and ϕ_0 are

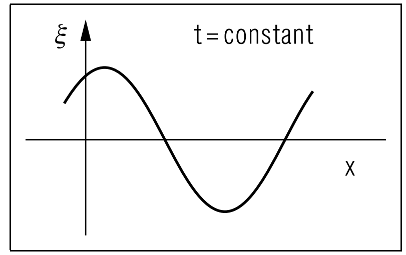


Figure 4. Snapshot of a harmonic wave function at some specified time t.

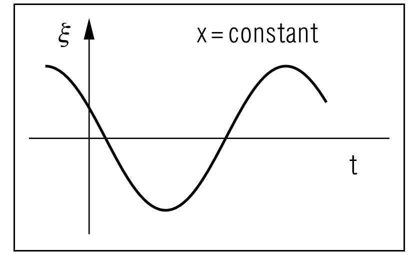


Figure 5. Plot of the same wave as in Fig. 4, but at a specified point x.

constants determined by the initial displacement and initial velocity of ξ at some point in space.

3b. Phase and Phase Constant. The argument of a harmonic function is called the "phase" of the wave, ϕ. For a one-dimensional traveling wave:

$$\phi(x,t) = \frac{2\pi}{\lambda}(x \pm vt) + \phi_0\,. \tag{3}$$

The phase describes the part of a complete wave oscillation that is occurring at a given place and time. The constant ϕ_0 is called the "phase constant" and is the value of ϕ at $x = 0$, $t = 0$. The units of the phase are radians when 2π occurs in Eq. (3). To express ϕ in degrees, replace 2π radians with its equivalent, 360°.

3c. Amplitude, Wavelength, and Wave Number. The maximum value that $\sin\phi$ or $\cos\phi$ may take is ± 1, so the maximum wave disturbance ξ is:

$$\xi_{\max} = \pm A\,. \tag{4}$$

This maximum value of ξ is called the "amplitude" of the wave. The point where $\xi = +A$ is typically called the "crest" of the wave and the point where $\xi = -A$ is called the "trough" of the wave.[2] The distance from crest to crest (or trough to trough) is called the "wavelength," the distance between points on the wave which have the same phase at the same instant of time. Figure 6 illustrates these wave dimensions for a sinusoidal wave. A useful expression involving the wavelength is the definition of a quantity

[2] Notice that the designation of a wave maximum as a crest or a trough is somewhat arbitrary since the maxima at $\xi = +A$ are identical to the maxima at $\xi = -A$.

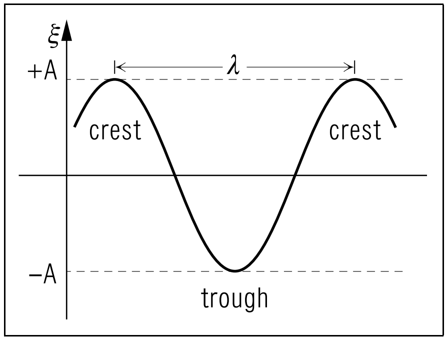


Figure 6. Amplitude and wavelength of a harmonic wave.

called the wave number, k, of a wave,

$$k = \frac{2\pi}{\lambda} \,. \tag{5}$$

Since wavelength has units of length and 2π radians is a dimensionless quantity, k has units of inverse length, usually m^{-1} or cm^{-1}. Using the wave number symbol, a one-dimensional sinusoidal wave function may be expressed as

$$\xi = A \sin\left[k(x \pm vt) + \phi_0\right] \,. \tag{6}$$

3d. Phase: Period, Frequency, Angular Frequency. There are a number of ways of writing the phase of a wave, depending on whether one uses period, frequency, or angular frequency. For a particle at a fixed point, undergoing simple harmonic motion, we can write the phase of its motion as

$$\phi = \left(\frac{2\pi}{T}\right) t + \phi_0 \,, \tag{7}$$

where T is the period of the motion, and ϕ_0 is the initial phase. Comparing Eqs. (7) and (3), we see that the phase of a one-dimensional harmonic wave may be written as

$$\phi = 2\pi \left(\frac{x}{\lambda} \pm \frac{t}{T}\right) + \phi_0 \,. \tag{8}$$

Using Eq. (8), the phase of a wave is easy to interpret: a change in position by one wavelength or a change in time by one period results in a change in phase of 2π radians (360°). The phase of a harmonic wave may also be expressed in terms of frequency or angular frequency. Using the relation between period and frequency

$$\nu = \frac{1}{T} \,, \tag{9}$$

the phase may be written as

$$\phi = 2\pi \left(\frac{x}{\lambda} \pm \nu t\right) + \phi_0 \,, \tag{10}$$

or using the relation between frequency and angular frequency

$$\omega = 2\pi\nu \tag{11}$$

and the definition of wave number, Eq. (5), the phase becomes

$$\phi = kx \pm \omega t + \phi_0 \,. \tag{12}$$

3e. Relations Among Traveling-Wave Descriptors. By comparing Eq. (3) and (8), a relation between the wave speed v, the wavelength λ, and the period T, may be determined to be:

$$v = \frac{\lambda}{T} \,. \tag{13}$$

This expression may be transformed into several equivalent forms by using the definition of frequency and angular frequency:

$$v = \lambda\nu \,, \tag{14}$$

and

$$v = \frac{\omega}{k} \,. \tag{15}$$

Using these relations between wave descriptors and their definitions, you should be able to transform between the several forms of the wave function we have encountered so far. These relations and the various forms of the harmonic wave function are summarized in Table 1. The variables used in Table 1 are listed in Table 2, although you should note that: (1) any part of a wave could be used in place of the word "crests"; and (2) the descriptions are only meant as reminders of the more complete descriptions given throughout the text.

Table 1. Useful wave relations and various one-dimensional harmonic wave functions. Remember that cosine functions may also be used as harmonic wave functions.

Wave Relations	One-Dimensional Wave Functions
$v = \dfrac{\lambda}{T}$	$\xi = A \sin\left[\dfrac{2\pi}{\lambda}(x \pm vt) + \phi_0\right]$
$v = \lambda\nu$	$\xi = A \sin\left[k\,(x \pm vt) + \phi_0\right]$
$v = \dfrac{\omega}{k}$	$\xi = A \sin\left[2\pi\left(\dfrac{x}{\lambda} \pm \dfrac{t}{T}\right) + \phi_0\right]$
$k = \dfrac{2\pi}{\lambda}, \nu = \dfrac{1}{T}$	$\xi = A \sin\left[2\pi\left(\dfrac{x}{\lambda} \pm \nu t\right) + \phi_0\right]$
$\omega = 2\pi\nu$	$\xi = A \sin\left[kx \pm \omega t + \phi_0\right]$

Table 2. Variables used in Table 1.

Variable	Brief Description
λ	wavelength: distance between successive crests at one time
T	period: time between successive crests at one place
ξ	wave function: the size of the wave at any time and place
A	amplitude: maximum value of the wave function
v	speed of each crest
t	the time at which the wave function is being described
ϕ_0	phase constant: the wave's phase at time zero, place zero
k	wave number: number of waves per unit length at one time
x	the place at which the wave function is being described
ω	angular frequency: 2π times the frequency
ν	frequency: rate at which crests go by at one place

4. The Equation of Wave Motion

4a. One-Dimensional Equation of Wave Motion . By applying Newton's second law and some forms of Hooke's law to the deformation ξ in an elastic medium, a differential equation of motion for ξ may be

derived.[3] If ξ is a one-dimensional traveling wave in the elastic medium, the differential equation of motion is found to be:

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \frac{\partial^2 \xi}{\partial x^2}\,. \tag{16}$$

This partial differential equation[4] is called the "characteristic equation" of wave motion in one dimension. If, by using Newton's second law, you find for a physical system that the equation of motion is of this form, then you know that wave motion can result. From this differential equation you can read the propagation speed of any wave that obeys it.

4b. Forms of the General Solution. Since the wave equation is a second order linear partial differential equation, the general solution of the wave equation consists of a linear combination of two linearly independent harmonic functions:

$$\xi(x,t) = f_1(x \pm vt) + f_2(x \pm vt)\,. \tag{17}$$

You should be able to verify Eq. (17) as a solution to Eq. (16), the wave equation, by direct substitution.

If the signs of the "vt" terms are the same in f_1 as in f_2, Eq. (17) represents a superposition of two waves traveling in the same direction.

If the signs of "vt" terms are opposite for the two functions in Eq. (17), we have the superposition of two waves traveling in opposite directions. With appropriate choices for boundary conditions, this particular solution to the wave equation is called a "standing wave" and it is a very important phenomenon in physics. It is treated elsewhere.[5]

4c. Restriction to Harmonic Waves. We can restrict Eq. (17) to a description of single-frequency harmonic waves by making one term a sine function and the other a cosine function (these functions are linearly independent):

$$\xi(x,t) = A\,\sin[k(x - vt)] + B\,\cos[k(x - vt)]\,. \tag{18}$$

The two amplitudes A and B complete the description of particular waves. That is, specifying values for them picks out a specific case from all possible waves with the specified frequency and velocity already specified in

[3]For several examples of this derivation, see "Sound Waves and Small Transverse Waves on a String" (MISN-0-202).

[4]For the meaning of "partial differential equation" and "partial derivative," as used in this module, see this module's Appendix.

[5]See "Standing Waves" (MISN-0-232) and "Standing Waves in Sheets of Materials" (MISN-0-233).

Eq. (18). For example, if the values of ξ and its first derivative with respect to time are known at some point in space and at some instant of time for the process at hand (usually at $x = 0$ and $t = 0$), then those values can be used to set A and B. As an alternative to Eq. (18), we can express ξ as a single sine or cosine function (more useful in certain situations):

$$\xi(x,t) = \xi_0 \sin[k(x - vt) + \phi_0]\,, \tag{19}$$

where Eqs. (18) and (19) are connected by:

$$\xi_0 = \left(A^2 + B^2\right)^{1/2} \text{ and } \phi_0 = \tan\left(\frac{B}{A}\right). \tag{20}$$

Either way we have two constants that must be established for any particular application.[6]

Acknowledgments

I would like to thank J. Kovacs, O. McHarris, and P. Signell for their contributions to an earlier version of this module. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **amplitude**: the maximum value of a wave function.
- **boundary conditions**: the values of the wave function and its first time-derivative, at some point in space, at some instant of time.
- **field**: a quantity that has a value at each point in space.
- **harmonic function**: a sinusoidal function; for example, $\sin(kx \pm \omega t)$ or $\cos(kx \pm \omega t)$.
- **partial derivative**: a derivative of an expression where only one quantity in the expression can vary.

[6]The requirement of two constants may not surprise you since the solution of a second order differential equation requires two sequential integrations, thus introducing two constants of integration. These two constants are called "boundary conditions" or "initial conditions," but both of those terms are misleading since the values used are not necessarily on any physical periphery nor are they necessarily values for the boundaries of the time interval being studied.

- **phase**: the argument of a harmonic wave function. The phase of a wave specifies what part of a complete oscillation or cycle the wave is producing at a given point in space and at a given time.
- **phase constant**: the value of the phase of a wave at time $t = 0$ at the origin of the relevant coordinate system.
- **sinusoidal wave**: a wave whose spatial profile at any given time is a sine function and which produces simple harmonic oscillations of the wave quantity ξ at any given point through which the wave passes.
- **traveling wave**: a periodic series of oscillations that propagate through space with a speed characteristic of the wave and the medium through which it travels.
- **wave equation**: a differential equation of motion whose solutions are mathematical representations of waves.
- **wave function**: a mathematical representation of a wave, and the solution to the wave equation.
- **wave number**: a quantity inversely proportional to the wavelength of a wave; symbolized by k. Note: $k = 2\pi/\lambda$.
- **wave speed**: the speed with which a wave propagates through space.
- **wavelength**: the distance in space between successive points of equal phase on a wave; symbolized by λ.

Partial Derivatives

a. Particles and Ordinary Derivatives. A "particle" is usually specified in part by its position, say x. The position of the particle usually changes with time so we write $x(t)$. Then the *rate* of change of the particle's position with respect to time is written $dx(t)/dt$, and that is its x-component of velocity. It is an example of the use of the ordinary derivative.

b. Fields: Temperature as an Example. A "field" is specified by its value at each space point (x) at each time (t). For example, the temperature of the air in this room can be written as $T(x,t)$: at a particular time t it has a value (in degrees) at each space point x as one moves along some straight line across the room. As time changes, the value at each space point along the line changes. Note that there are an uncountable infinity of points along the line, at each of which the temperature can be specified at any one time. This is in contrast to a "particle," for which there is only one position at any one time.

c. A Mental Exercise. In your mind, go through the process of plotting an ordinary two-dimensional graph showing the temperature at all points x for a fixed time t. That is, plot $T(x)$ for a single time t, a "snapshot" of the temperature field. Now imagine plotting a separate graph showing the temperature at a single point (that is, at a single x value), as a function of time. Think about how the measurements would be made in each case.

d. Fields and Partial Derivatives: an Example. If we want to know the rate of change of temperature, T, with position along a line, x, *at a fixed time* t, we must take the derivative of T with respect to x while holding t fixed. This process of holding one variable fixed while taking the derivative with respect to another variable is called "taking a partial derivative." It is written with ∂ symbols replacing the usual d symbols in the derivative. Here are some examples of taking first and second partial derivatives of a particular function $f(x,t)$:

$$\begin{array}{ll} & f = (x+vt)^2 \\ \partial f/\partial x = 2(x+vt) & \partial f/\partial t = 2v(x+vt) \\ \partial^2 f/\partial x^2 = 2 & \partial^2 f/\partial t^2 = 2v^2 \end{array}$$

PROBLEM SUPPLEMENT

Note: Problems 8, 9, and 10 also occur in this module's *Model Exam*.

1. Given the function:

$$\xi(x,t) = A_1 \cos k_1(x+vt) - A_2 \sin k_2(x-vt)\,.$$

Determine whether this a solution to the wave equation, *Help: [S-1]*

$$v^2 \frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2}\,.$$

2. Let $\xi(x,t) = A\sin(\omega t + kx + \phi_0)$ where ϕ_0 is the phase constant. This wave is traveling in the negative x-direction at the speed of sound in air, 330 m/s.

   a. Determine whether $\xi(x,t)$ satisfies the wave equation quoted in Problem 1.

   b. If $A = 6.0\,\text{cm}$, $\xi(0,0) = 5.196\,\text{cm}$ and

$$\dot{\xi}(0,0) \equiv \left.\frac{\partial \xi}{\partial t}\right|_{\substack{x=0\\t=0}} = +94.2\,\text{m/s}\,,$$

   find ϕ_0, ω, k, T, ν, and λ.

   c. Sketch $\xi(x)$ at $t=0$.

3. A certain one-dimensional wave is observed at a certain instant of time to be described by:

$$\xi(x,t_1) = (1.3\,\text{m})\sin[(1.2\,\text{m}^{-1})x + 16\pi]$$

and 12 seconds later by:

$$\xi(x,t_1+12\,\text{s}) = (1.3\,\text{m})\sin[(1.2\,\text{m}^{-1})x + 28\pi].$$

Determine this wave's: (a) amplitude; (b) wavelength; (c) frequency (in hertz); (d) speed; and (e) direction of travel.

4. A function $y(x)$ consists of: (1) a straight line that increases from its value of zero at position $(x_0 - A)$ to its maximum value of M at position x_0; then (2) another straight line from this maximum value of M to be the value zero at position $(x_0 + A)$. Everywhere else, $y(x)$ is zero. Thus the function is a triangle in the x-y plane joining points $(x_0 - A, 0)$, (x_0, M) and $(x_0 + A, 0)$.

   a. Sketch the function $y(x)$.

   b. Sketch the function $y(x + A)$.

   c. Sketch the function $y(x - 2A)$.

5. Show that $\xi = \xi_0 \sin(kx - \omega t)$ may be written in the alternative forms:

   a. $\xi = \xi_0 \sin[k(x - vt)]$

   b. $\xi = \xi_0 \sin\left[\omega\left(\frac{x}{v} - t\right)\right]$

   c. $\xi = \xi_0 \sin\left[2\pi\left(\frac{x}{\lambda} - \nu t\right)\right]$

   d. $\xi = \xi_0 \sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$

6. A one-dimensional sinusoidal wave, of wavelength 2 m, travels along the x-axis (in the positive x-direction). If its amplitude is 0.5 m and it has a period of $T = 0.5\,\mathrm{s}$:

   a. Write down an appropriate wave function to represent this wave.

   b. If the displacement of the wave is 0.2 m at $x = 0$, $t = 0$, and $\dot{\xi}(0,0) = +5.76\,\mathrm{m/s}$, find the phase constant.

7. A one-dimensional sinusoidal wave moves along the x-axis. The displacement at two points, $x_1 = 0$ and $x_2 = 2.0\,\mathrm{cm}$, is observed as a function of time:

$$\begin{aligned}\xi(x_1, t) &= (0.02\,\mathrm{cm})\sin[(3\pi\,\mathrm{s}^{-1})t] \\ \xi(x_2, t) &= (0.02\,\mathrm{cm})\sin[(3\pi\,\mathrm{s}^{-1})t + \frac{\pi}{2}]\end{aligned}$$

   a. What are the amplitude, frequency, and wavelength of this wave? *Help: [S-10]*

   b. In which direction and with what speed does the wave travel?

8. Verify whether or not each of the following functions is a solution to the one-dimensional wave equation:

   a. $\xi = \xi_0 \cos(\pi t)$

   b. $\xi = \xi_0 \sin M(x + 4vt)$ where M is a constant

   c. $\xi = Y(x - vt)$

9. A one-dimensional sinusoidal wave is traveling along the x-axis in the negative x-direction. It can be represented by:

$$\xi(x,t) = A\cos[\frac{2\pi}{\lambda}F(x,t)].$$

   The wave's frequency is 10 Hz and its wavelength is λ. Write down an appropriate function $F(x,t)$ which gives this wave the properties listed above.

10. The displacements at two points in space are observed as a wave $\xi(x,t)$ passes by. At the points $x_1 = 0.5\,\mathrm{m}$ and $x_2 = 2.5\,\mathrm{m}$ the displacement from equilibrium is observed as a function of time. These are found to be: $\xi(0.5\,\mathrm{m}, t) = (1.5 \times 10^{-4}\,\mathrm{m})\sin[(6\pi\,\mathrm{s}^{-1})t]$
$\xi(2.5\,\mathrm{m}, t) = (1.5 \times 10^{-4}\,\mathrm{m})\sin[(6\pi\,\mathrm{s}^{-1})t + 2\pi/3]$

    a. What is the amplitude of this wave?

    b. What is the frequency of this wave in hertz?

    c. What is the wavelength?

    d. What is the speed with which this wave travels?

    e. What way is the wave traveling?

    f. What is the time rate of displacement at the point x_1 at times $t = 0$ and $t = 0.25\,\mathrm{s}$?

Brief Answers:

1. Yes *Help: [S-1]*

2. A traveling wave:

   a. Yes, if $\omega^2 = v^2k^2$ (see Problem 1).

   b. $\phi_0 = \pi/3\,\mathrm{radians} = 60^\circ$ *Help: [S-3]*

      $\omega = 3.14 \times 10^3\,\mathrm{s}^{-1}$ *Help: [S-2]*

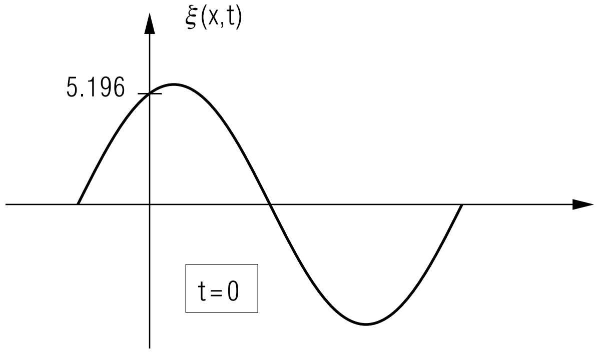
      $k = 9.5\,\mathrm{m}^{-1}$

$T = 2 \times 10^{-3}\,\mathrm{s}$

$\nu = 500\,\mathrm{Hz}$

$\lambda = 0.66\,\mathrm{m}$

c. $\xi(x,t)$ at $t = 0$:



3. A wave described at two times:

   a. $A = 1.3\,\mathrm{m}$ *Help: [S-5]*

   b. $\lambda = 5.24\,\mathrm{m}$ *Help: [S-6]*

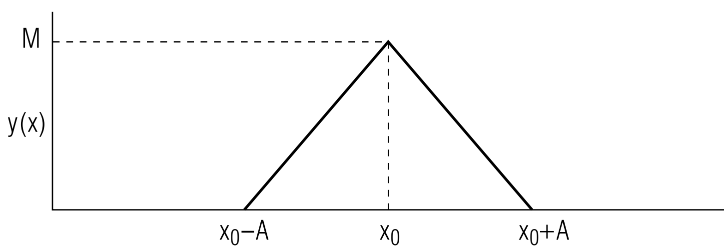
   c. $\nu = 0.50\,\mathrm{Hz}$ *Help: [S-4]*

   d. $v = 2.62\,\mathrm{m\,s^{-1}}$ *Help: [S-7]*

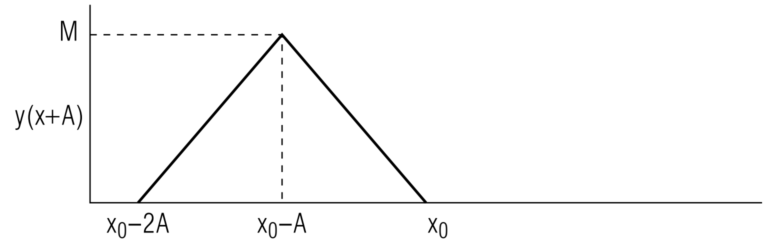
   e. $-\hat{x}$ direction *Help: [S-8]*

4. $y(x_0)$ is the maximum value of $y(x)$.

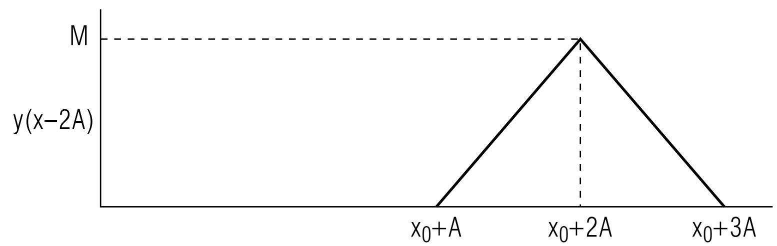
   a. $y(x)$ has its maximum value when $x = x_0$:



   b. $y(x + A)$ has its maximum value when $x + A = x_0$:



   c. $y(x - 2A)$ has its maximum value when $x - 2A = x_0$:



5. $k = \omega/v$; $kv = \omega$; $(\omega/v) = (2\pi/\lambda)$; $\omega = 2\pi\nu$; $\nu = 1/T$.

6. Determining the functional form.

   a. $\xi = (0.5\,\mathrm{m})\sin\left[\left(\pi\,\mathrm{m}^{-1}\right)x - \left(4\pi\,\mathrm{s}^{-1}\right)t + \phi_0\right]$,

      or

      $\xi = (0.5\,\mathrm{m})\cos\left[\left(\pi\,\mathrm{m}^{-1}\right)x - \left(4\pi\,\mathrm{s}^{-1}\right)t + \phi_0\right]$.

   b. $\phi_0 = 156.4^\circ$ if the sine function is used;
      $\phi_0 = 66.4^\circ$ if the cosine function is used. *Help: [S-9]*

7. A wave specified at two times:

   a. $A = 0.02\,\mathrm{cm}$; $\nu = 1.5\,/\mathrm{s}$; $\lambda = 8.0\,\mathrm{cm}$ *Help: [S-10]*

   b. The wave moves in the negative x direction with speed $v = 12\,\mathrm{cm/s}$.

8. Only (c) satisfies this equation.

9. $F(x,t) = x + (10\,\mathrm{Hz})\lambda t$

10. Displacements at two points:

    a. $1.5 \times 10^{-4}\,\mathrm{m}$

b. 3 Hz

c. 6 m

d. 18 m/s.

e. Toward the negative x-direction

f. $9\pi \times 10^{-4}$ m/s and zero respectively.

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-Problem 1)*

Taking the appropriate partial derivatives,

$$\begin{aligned}
\frac{\partial \xi}{\partial t} &= -k_1 v A_1 \sin\left[k_1(x+vt)\right] + k_2 v A_2 \cos\left[k_2(x-vt)\right] \\
\Rightarrow \frac{\partial^2 \xi}{\partial t^2} &= -k_1^2 v^2 A_1 \cos\left[k_1(x+vt)\right] + k_2^2 v^2 A_2 \sin\left[k_2(x-vt)\right] \\
\frac{\partial \xi}{\partial x} &= -k_1 A_1 \sin\left[k_1(x+vt)\right] - k_2 A_2 \cos\left[k_2(x-vt)\right] \\
\Rightarrow \frac{\partial^2 \xi}{\partial x^2} &= -k_1^2 A_1 \cos\left[k_1(x+vt)\right] + k_2^2 A_2 \sin\left[k_2(x-vt)\right]
\end{aligned}$$

Comparing the two equations marked $\Rightarrow$, it is clear that v^2 times the lower one equals the upper one. Thus ξ is a solution to the wave equation. However, if the velocity v in the two terms had not been the same, ξ would not have been a solution. Note that the first term represents a wave of amplitude A_1 and wavelength $2\pi/k_1$ traveling to the left, while the second term represents a wave of amplitude A_2 and wavelength $2\pi/k_2$ traveling to the right. These two waves are traveling through the same space points at the same time.

S-2 *(from PS-Problem 2b)*

$$\omega = \frac{\dot{\xi}(0,0)}{A\cos\phi_0} = \frac{94.2\,\mathrm{m/s}}{(6.0\,\mathrm{cm})(1/2)}$$

S-3 *(from PS-Problem 2b)*

$\xi(0,0) = A\sin(0+0+\phi_0) = A\sin\phi_0 = 5.196\,\mathrm{cm}$

$\sin\phi_0 = \dfrac{\xi(0,0)}{A} = \dfrac{5.196}{6.0} = 0.866,$

so: $\phi_0 = \sin^{-1}(0.866) = 60^\circ$ $(\pi/3\,\mathrm{radians})$ or 120° $(2\pi/3\,\mathrm{radians})$

To choose the correct value of ϕ_0, information from $\dot{\xi}(0,0)$ must be used:

$\dot{\xi}(x,t) \equiv \partial\xi/\partial t = \omega A\cos(kx+\omega t+\phi_0)$

$\dot{\xi}(0,0) = \omega A\cos\phi_0 = +94.2\,\mathrm{m/s}$

$\cos\pi/3 = +1/2$, $\cos(2\pi/3) = -1/2$.

Since $\dot{\xi}(0,0)$, ω, and A are all positive, $\cos\phi_0$ must be positive as well, so $2\pi/3$ is rejected as a possible value for ϕ_0, i.e. $\phi_0 = \pi/3$.

S-4 *(from PS-Problem 3c)*

Rule: $\Delta(\text{phase}) = 2\pi\left[\dfrac{\Delta x}{\lambda} \pm \dfrac{\Delta t}{T}\right]$.

For this case,
phase #1= $(1.2\,\mathrm{m}^{-1})x + 16\pi$
phase #2= $(1.2\,\mathrm{m}^{-1})x + 28\pi$
$\Delta(\text{phase}) = 12\pi$

but $\Delta x = 0$, hence: $2\pi\left(\dfrac{\Delta t}{T}\right) = 12\pi$, and $\Delta t = 12\,\mathrm{s}$.

S-5 *(from PS-Problem 3a)*

Just look at either $\xi(x,t_1)$ or $\xi(x,t_2)$.

S-6 *(from PS-Problem 3b)*

See Problem 1. $\lambda = 2\pi/(1.2\,\mathrm{m}^{-1})$.

S-7 *(from PS-Problem 3d)*

Compare to the general form of the wave equation to get:
$kv\Delta t = \Delta\phi \Rightarrow v = 12\pi/(1.2\,\mathrm{m}^{-1}\ 12\,\mathrm{s})$

S-8 *(from PS-Problem 3e)*

Read and understand this module's *text*.

S-9 *(from PS-Problem 6b)*

Note 1: -203.6° is just as good an answer as 156.4°.
Note 2: Electronic calculators face an ambiguity in giving an answer for an inverse trigonometric function. For example, $\sin(5.74^\circ) = \sin(174.26^\circ) = 0.1$. Therefore $\sin^{-1}(0.1)$ could be either 5.74° or 174.26°: both are valid mathematical answers to the inverse sine problem, taken in isolation. You must decide which is the right answer by examining other aspects of the problem at hand.

S-10 *(from PS-Problem 7)*

Compare to the general form of the wave equation to get:
$(2\pi/\lambda)\Delta x = \Delta\phi \Rightarrow \lambda = 2\pi(2.0\,\mathrm{cm})/(\pi/2)$

MODEL EXAM

1. See Output Skills K1-K3 in this module's *ID Sheet*.

2. Verify whether or not each of the following functions is a solution to the one-dimensional wave equation:

   a. $\xi = \xi_0 \cos(\pi t)$

   b. $\xi = \xi_0 \sin M(x + 4vt)$ where M is a constant

   c. $\xi = Y(x - vt)$

3. A one-dimensional sinusoidal wave is traveling along the x-axis in the negative x-direction. It can be represented by:

$$\xi(x,t) = A \cos \left[\frac{2\pi}{\lambda} F(x,t) \right] .$$

   The wave's frequency is 10 Hz and its wavelength is λ. Write down an appropriate function $F(x,t)$ which gives this wave the properties listed above.

4. The displacements $\xi(x,t)$ at points in space are observed as a wave passes by. At the points $x_1 = 0.5\,\mathrm{m}$ and $x_2 = 2.5\,\mathrm{m}$ the displacements from equilibrium, ξ, are found to be (as functions of time):

$$\xi(0.5\,\mathrm{m}, t) = (1.5 \times 10^{-4}\,\mathrm{m}) \sin[(6\pi\,\mathrm{s}^{-1})t]$$
$$\xi(2.5\,\mathrm{m}, t) = (1.5 \times 10^{-4}\,\mathrm{m}) \sin[(6\pi\,\mathrm{s}^{-1})t + 2\pi/3]$$

   a. What is the amplitude of this wave?

   b. What is the frequency of this wave in hertz?

   c. What is the wavelength?

   d. What is the speed with which this wave travels?

   e. What direction is the wave traveling?

   f. What is the time-rate of displacement at the point x_1 at times $t = 0$ and $t = 0.25\,\mathrm{s}$?

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 8.

3. See this module's *Problem Supplement*, problem 9.

4. See this module's *Problem Supplement*, problem 10.