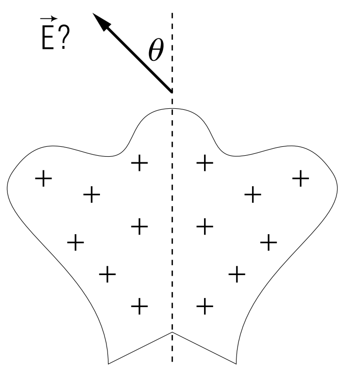
MISN-0-153

ELECTRIC FIELDS FROM SYMMETRIC CHARGE DISTRIBUTIONS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

ELECTRIC FIELDS FROM
SYMMETRIC CHARGE DISTRIBUTIONS

by
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Input Skills:

1. Vocabulary: electric field direction and magnitude, (MISN-0-115); linear charge density, surface charge density, volume charge density, charge distribution (MISN-0-147); invariant, homogeneous, rotational invariance, bisector, longitudinal axis (Glossary).
2. State the formula for the electric field of a point charge, explaining all symbols (MISN-0-115).

Output Skills (Knowledge):

K1. Vocabulary: equi-magnitude surface (for an electric field), cylindrical symmetry, planar symmetry, spherical symmetry.

Output Skills (Rule Application):

R1. Given a charge distribution, determine whether it does or does not have spherical, cylindrical or planar symmetry.

R2. Given a charge distribution having spherical, cylindrical or planar symmetry, state the rules for determining the directions and equi-magnitude surfaces of the electric field, using the symmetry of the charge distribution, and show the correctness of those rules.

R3. Given a charge distribution having spherical, cylindrical or planar symmetry, demonstrate use of the rules for determining the directions and equi-magnitude surfaces of the electric field near the charge distribution.

Post-Options:

1. "Gauss's Law and Spherically Distributed Charges" (MISN-0-132).
2. "Gauss's Law Applied to Cylindrical and Planar Charge Distributions" (MISN-0-133).

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1. Introduction

In this module we describe two very simple rules for finding useful parts of electric fields. These rules enable us to bypass complex mathematical calculations and obtain answers directly.[1] Although the rules are powerful, the other side of the coin is that they are useful only when the associated electric charges are distributed in simple geometrical shapes. Fortunately, the charges in most modern electronic components are distributed in just such simple shapes. For them the rules can be applied almost effortlessly. One rule produces the fields' directions and the other the surfaces on which field magnitudes are constant. The two rules are independent of each other so we discuss them separately.

2. The Field-Direction Rule

2a. Overview. The "field direction" rule allows us to find the electric field direction at points along certain lines in space near certain kinds of charge distributions. Given a specific charge distribution and asked for the direction of the nearby electric field, we first identify lines of symmetry in the charge distribution, such as the dashed vertical line in Fig. 1, a line of left-right symmetry. Once we have identified a line of symmetry, we can find the field direction at all points along that line.

2b. Identifying an Appropriate Line. Any line along which we apply the "field direction" rule must be an axis of "180°-rotation symmetry." That is, if we rotate our charge distribution about such an axis, through 180° (one half of a complete turn), then the new appearance of the charge distribution must be indistinguishable from its appearance before the rotation. In Fig. 1, see that the vertical dashed line is just such an axis: mentally make a 180° rotation about the axis and see that the resulting charge distribution is indistinguishable from the one we had before the rotation. Also see that there is no additional axis of 180°-rotation

[1] For applications see "Electrostatic Field Due To Continuous Charge Distributions: Relation to Potential" (MISN-0-148).

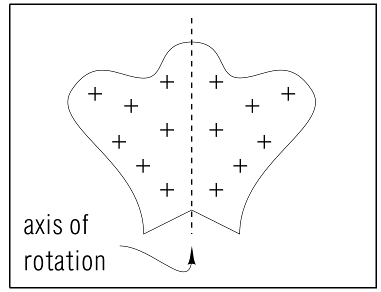

Figure 1. A charge distribution that is the same before and after a 180° rotation about the axis shown.

symmetry for that charge distribution.

2c. The Rule. Here is the "field direction" rule:

> *Find an axis about which the charge distribution is invariant to 180° rotations. At all points on that axis the direction of the electric field is along the axis.*

Note that this rule gives the field direction only along axes for which the charge distribution is exactly the same before and after a 180° rotation (see Fig. 1). Where two such axes cross, the electric field must be zero.

2d. Proving the Rule. The "field direction" rule is easily shown to be correct by demonstrating that any deviation from it is false. Look at Fig. 2a where the rule requires that $\theta = 0$. Suppose we violate the rule and take $\theta \neq 0$, as shown in the figure. Now keep the axis fixed and mentally rotate the charge distribution through 180° about that axis. Because the charge is what produces the electric field, the field must rotate along with the charge, just as though it were somehow rigidly attached to it. After the 180° rotation is over, the charge distribution is exactly the same as it was before the rotation (because we are only dealing with cases where the charge distribution has this invariance). Since the charge distribution is the same as before the rotation, the electric field that the charge produces must also be the same, but we see that the direction of the field did change! Thus our electric field with $\theta \neq 0$ must be incorrect. Compare Figs. 2a and 2b and see that the only way the field could be the same before and after the rotation is if $\theta = 0$. This means that the field must be along the axis, just as the rule states. Thus the rule has been proven.

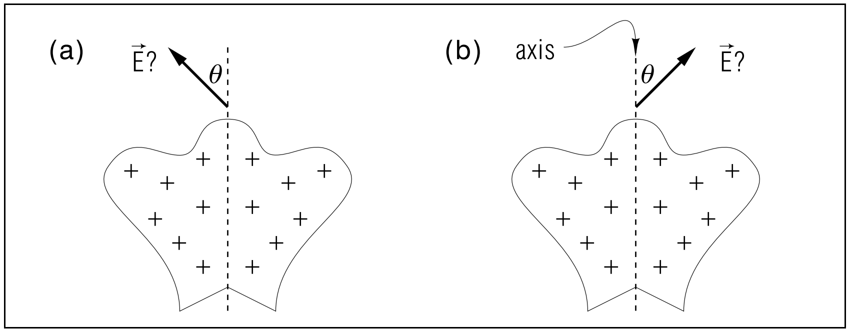

Figure 2. The result of assuming the electric field direction does not lie along the 180° symmetry axis of its charge distribution: (a) before; and (b) after a 180° rotation about the axis.

3. Field Directions

3a. A Uniform Sphere. For a spherical volume with a uniform charge distribution, any straight line through the center of the sphere is an axis of rotational invariance (see Fig. 3). Try such a rotation in your head and see that the charge distribution stays precisely the same as the sphere is rotated.

For a uniform sphere of charge, the "field direction" rule says that the electric field direction is radial (with respect to the center of the sphere) at all points in space. This is because any point in space is on a radial line from the center of the sphere. Since any line through the center of the sphere is a line of rotational symmetry, the field direction rule says that the electric field direction is along the line and hence is radial with respect to the center of the sphere.

Note that a point charge is also invariant under rotations about any axis that goes through it, so it is a special case of the spherical distribution.

3b. A Uniform Cylindrical Surface. A charge distribution that is uniform over a cylindrical surface has a single axis plus a separate plane of axes along which we can apply the field direction rule. The single axis is down the center of the cylinder: it is certainly one about which the charge distribution is rotationally invariant (see Fig. 4). Imagine rotating the charge distribution about that longitudinal axis and seeing that the appearance of the charge distribution does not change.

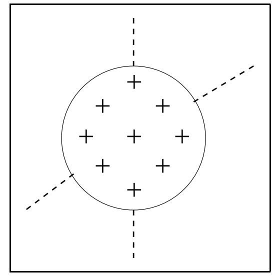

Figure 3. Two axes of rotational symmetry for a uniform sphere of charge.

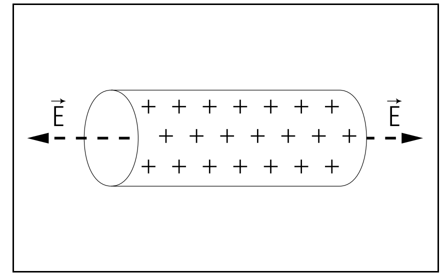

Figure 4. An axis of rotational symmetry for a cylindrical distribution of charge.

Since the single axis down the center of the cylinder is a line of rotational symmetry, the rule says that at any point on that axis the electric field direction is along the axis (see Fig. 4). The actual direction along the axis is governed by the usual rule: "Away from positive charge, toward negative charge."

In addition to the longitudinal axis, there are an infinite number of other axes of 180°-rotation invariance, all lying in the plane that bisects the cylinder into two equal cylinders: two such axes are shown in Fig. 5. Imagine other axes lying in the same plane as those two, and mentally apply the rule to each of your imagined axes.

We conclude that the field direction is along the longitudinal axis of the cylinder at all points on that axis, and is radial at all points lying in the plane that bisects the cylinder.

3c. An Infinitely Long Cylinder. For charges distributed uniformly along an infinitely long cylinder, the electric field is radial at all points in space. Look at Fig. 5 and imagine that the cylinder stretches to infinity to both left and right. Look at the two arrow-marked axes near the center of the sketch: these are axes of 180°-rotation symmetry. Now imagine keeping these axes where they are but moving the cylinder of charge a short distance to the right. Because the ends are an infinite distance away, the distribution of charge looks exactly the way it did before it was moved. Since the direction of the electric field was along the original axes, it must also be along the radial axes at their new positions on the cylinder. The distance they were moved could have been any finite amount, so the electric field outside the cylinder must be radial everywhere (see Fig. 6).

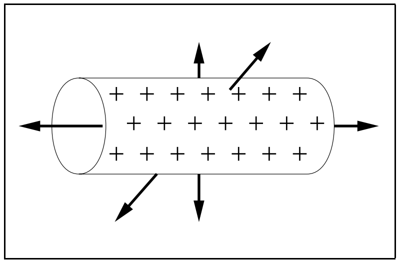

Figure 5. Three axes of 180°-rotation symmetry for a cylindrical charge distribution.

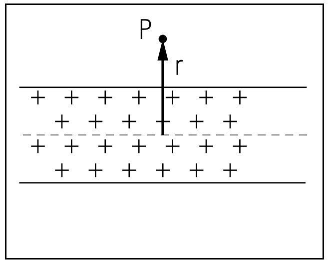

Figure 6. A point P at which we wish to know the direction of the electric field from an infinitely long cylinder of charge.

3d. Near A Finite Cylinder. For a charge distribution in the shape of a long thin finite cylinder, the electric field can be taken to be radial to the extent that the finite cylinder approximates an infinitely long one. In practice, this means that the point at which we determine the field direction must be much closer to the nearest charges than to either end of the cylinder ($r \ll$R in Fig. 8). To see that the field at such a close point is very close to being radial, note that an exactly radial field will be produced by a cylinder whose ends are equidistant from the field point (the unshaded part of the cylinder in Fig. 8). Then only the shaded part of the cylinder in Fig. 8 will produce a non-radial part to the field at P. However, this will be of little consequence for the field at P since the closer charges make a greater contribution to the field than do far-away charges: this is because the field falls off as the inverse square of the distance from individual charges. Of course the field direction will only be precisely radial at all field points of interest if the cylinder is infinitely long, making the ends an infinite distance away from our field points.

3e. Infinite Planes and Finite Planes. The direction of the electric field from an infinite plane (a flat sheet) of charge is, at any point in space, along the straight line that both goes through the point and is normal to the plane of the charge. This is easily seen: First, mentally construct a rotation axis that is normal to the sheet and that passes through the field point where the direction is desired (P in Fig. 9, although only a finite sheet is pictured). Then mentally see that the charge is invariant to rotations about the axis. That means that the field must be along the axis.

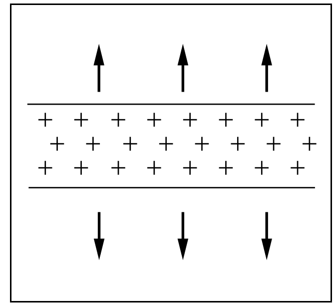

Figure 7. Electric field directions at various points outside an infinitely long cylinder of charge.

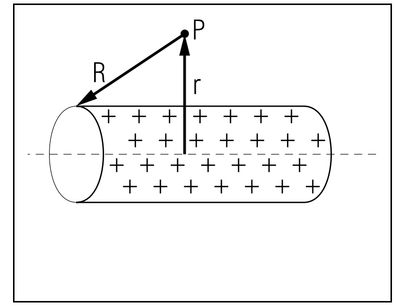

Figure 8. A point P at which we wish to know the direction of the electric field from a finite cylinder of charge.

For a finite rectangular sheet of constant density charge, the field will be exactly along the central axis normal to the sheet (see Fig. 9), and approximately along normals at space points near the sheet far from its edges. That is, if the sheet's nearby charges are very much closer to the field point than are charges along the sheet's edges, then the sheet's edges might as well be extended to infinity in all directions and the field will be invariant to rotations about any line that is normal to the sheet.

4. Equi-Magnitude Surfaces for *E*

4a. Definition. A space point's "electric field equi-magnitude surface": (i) includes the space point in question; and (ii) is a surface over which the magnitude of the electric field is constant.

For example, suppose we are at a point P that is at a distance r from a point charge Q (see Fig. 10). Then the magnitude of the electric

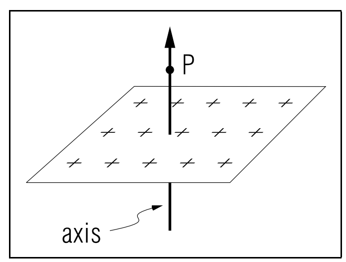

Figure 9. An axis of 180°-rotation symmetry for a rectangular sheet of constant-density charge.

field at P is $E = |\vec{E}| = k_e Q/(r^2)$. In fact, the electric field has this same magnitude at any point on the spherical surface of radius r that is centered on the point charge. Of course this surface includes the point P (see Fig. 10). Since the spherical surface satisfies the two conditions, it is indeed the equi-magnitude surface for the point P in the field of the point charge Q.

4b. The Rule. The rule for finding the equi-magnitude surface for a space point is this: starting at the space point, move along any trajectory for which the field's charge distribution always has the same appearance and is the same "distance away." The collection of all such trajectories defines the space point's equi-magnitude surface. This process is carried out mentally and is only feasible for several very simple distributions of charge.

4c. Example: A Point Charge. As an example of the "equi-magnitude surface," consider the space point P that is located a distance r from the point charge Q in Fig. 10. Suppose we move away from P but stay on the spherical surface that is centered on the charge and is at radius r. Then neither the point charge's apparent orientation nor its "distance away" will change. Since these quantities are all that can cause the magnitude of the charge's electric field to change, and these quantities stay constant, that magnitude must stay constant along the trajectory. The collection of all such trajectories defines the equi-magnitude surface.

4d. Example: Infinite Cylindrical Surface. For an infinitely long cylinder of charge, application of the "surface" rule shows that the magnitude of the electric field will be constant on any cylindrical surface concentric with it. One such a surface is labeled EMS in Fig. 11. We can also apply the rule to the case of two concentric cylindrical surfaces of charge, with or without different charge densities; any surface that is cylindrical and concentric with these surfaces is an equi-magnitude surface.

4e. General Results for Three Symmetries. We here catalog the result of applying the equi-magnitude surface rule to any charge distribution having one of our three symmetries. For some particular space point of interest, the surface depends on the symmetry of the charge distribution in this fashion:

1. for a charge distribution having spherical symmetry, it is the spherical surface that is centered on the center of spherical symmetry and which in addition goes through the space point;

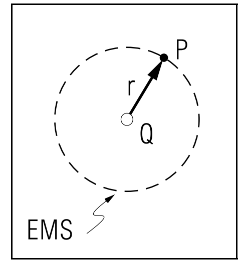

Figure 10. The equi-magnitude electric-field surface, EMS, for point P and charge Q.

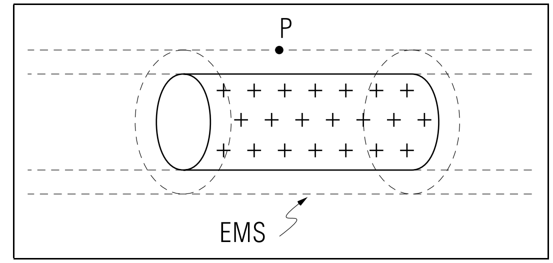

Figure 11. The surface, labeled EMS, through point P, over which the electric field has constant magnitude. The charge is uniformly distributed on an infinitely long cylindrical surface.

2. for a charge distribution having cylindrical symmetry, it is the cylindrical surface that is concentric with the axis of cylindrical symmetry and which in addition goes through the space point; and

3. for a charge distribution having planar symmetry, the electric field is constant over all space points so any surface is an equi-magnitude surface.

For non-infinite charge distributions, items 2 and 3 above will be good approximations as long as the distance to the edges is much larger than the distance to the nearest charges, just as was the case with the "direction" rule.

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University. Jerszy Borysowicz contributed a good wording for the physics related to Fig. 8.

Glossary

- **bisect**: to cut in half.

- **cylindrical symmetry**: a property of an object which means that its physical properties are invariant to rotations about an axis of symmetry and to translations along that axis.
- **electric field equi-magnitude surface**: a mathematical surface on which the electric field, due to some specified charge distribution, is constant.
- **invariant**: (adj.): property of not varying under some transformation or operation. The transformation or operation must be specified.
- **longitudinal**: for a long thin object, in the "long" direction.
- **planar symmetry**: a property of an object which means that its physical properties are invariant to translations in any direction in the symmetry plane. Any axis normal to the plane is an axis of rotational symmetry.
- **spherical symmetry**: a property of an object which means that its physical properties are invariant to all rotations about the center of symmetry.
- **translation**: a movement of an object or a coordinate system a finite distance along a straight line.

PROBLEM SUPPLEMENT

1. For a circle of charge, demonstrate the radial character of the electric field through direct use of the invariance of the charge distribution to 180° rotations.

Use the rules developed in the text to determine the electric field direction and equi-magnitude surfaces for charge distributions in the shapes of:

2. a point charge;
3. a spherical shell (a hollow sphere);
4. two concentric spherical shells;
5. a rectangular sheet (direction only);
6. two parallel infinite planes.

Brief Answers:

1. See text Sect. 2b for an analogous case.
2. (also 3, 4) With respect to the charge: radial; concentric spheres.
5. Along the three axes of 180°-rotation symmetry. Nothing can be said elsewhere.
6. With respect to the charge planes: normal; other parallel planes.

MODEL EXAM

1. Define . . . (See Output Skill K1 in this module's *ID Sheet*).
2. See Output Skills R1-R3 in this module's *ID Sheet*.

Brief Answers:

1. See this module's *text*.
2. Advice: (1) read Output Skills R1-R3; (2) understand all of the *Text*; (3) get to the point where you can work the problems in the *Problem Supplement* without looking back at the *text*: and (4) again read Output Skills R1, R2, and R3, making sure you can do them.