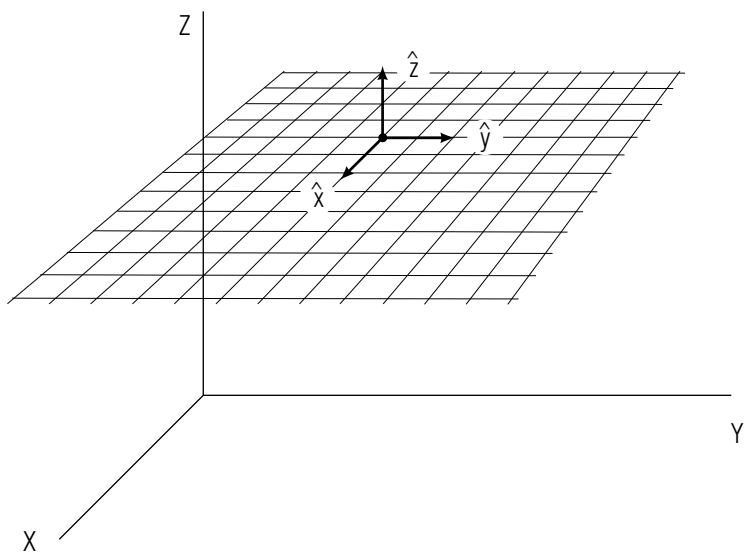PHYSNET

MISN-0-148

ELECTRIC FIELD AND POTENTIAL OF CONTINUOUS CHARGE DISTRIBUTIONS

ELECTRIC FIELD AND POTENTIAL OF CONTINUOUS
CHARGE DISTRIBUTIONS

by

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Title: **Electric Field and Potential of Continuous Charge Distributions**

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Input Skills:

1. Vocabulary: continuous charge distribution (MISN-0-147), electrostatic potential (MISN-0-116), gravitational potential (MISN-0-108), line charge (MISN-0-133), partial derivative (MISN-0-201).
2. Given the continuous function which describes the distribution of a system of charge, calculate the electric potential at any point in space (MISN-0-147).
3. Given the function that gives the electric field in a given region of space, determine the electric potential at points in that region (MISN-0-147).

Output Skills (Knowledge):

K1. Vocabulary: directional derivative, equipotential surface, gradient.

Output Skills (Problem Solving):

S1. Given the distribution of charge in a certain region of space, calculate directly the function that gives the electric field at all points in space due to this charge distribution.

S2. Given the function that specifies the electric potential at points in space, determine the function which describes the electric field vector at those same space points.

External Resources (Required):

1. Alonso and Finn, *Physics*, Addison-Wesley (1970).
2. E.M. Purcell, *Electricity and Magnetism*, Berkeley Physics Course-Vol 2, McGraw-Hill (1965).

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1. Introduction

If you can calculate the electric field at all points due to a given charge distribution, you will be in a position to determine completely the motion of any charged particle in the field of this distribution. Elsewhere the techniques for finding the electric potential due to a given distribution are developed.[1] In this module the procedure will be developed for getting the electric field directly from the given charge distribution as well.[2]

2. Suggested Procedure

Read sections 1.8 and 1.12 of Purcell.[3]

Note that $\epsilon_0 \equiv 1/(4\pi k_e)$ and $\mu_0 \equiv 4\pi k_m$.

Also read sections 2.3, 2.4, 2.5, and 2.6 which are appropriate for Output Skill S2.

Review section 16.9 of AF,[4] omitting nothing except Example 16.7, which is discussed elsewhere.[5]

Study what is summarized in Table 16.2 on page 351, especially the column on the right.

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[1] See "Electrostatic Potential Due to a Continuous Charge Distribution" (MISN-0-147).
[2] If the given charge distribution has associated with it some symmetry, then Gauss's Law provides a powerful technique for easily and directly getting the electric field due to this charge distribution. The series of modules MISN-0-153, and MISN-0-132 to MISN-0-135, deal with this.
[3] *Electricity and Magnetism*, E. M. Purcell (Berkeley Physics Course - Vol 2, McGraw - Hill, 1965). For access, see this module's *Local Guide*.
[4] M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading (1970). For access, see this module's *Local Guide*.
[5] "Electric Dipoles" (MISN-0-120).

Work problems 16.25 and 16.26.

Do these by starting with the solutions you got[6] for the potential functions associated with these charge distributions and finding the $\vec{E}$ field by taking the gradient of V. The answers to the two AF problems are given in this module's *Problem Supplement*. Also work the problems in the *Problem Supplement*.

3. Comments

We use the expression $\vec{E} = -\vec{\nabla}V$ throughout this subject. If you are totally unfamiliar with the $\vec{\nabla}$ operator, see Sect. 4. Below we present a short review.

The $\vec{\nabla}$ operator is called the "gradient operator" and used to be written **grad**. Note that it is *not* written Δ.

The relationship $\vec{E} = -\vec{\nabla}V$ is a vector relationship and in Cartesian coordinates:

$$\vec{\nabla}V \equiv \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z},$$

so that

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_y = -\frac{\partial V}{\partial y}, \qquad E_z = -\frac{\partial V}{\partial z},$$

giving the three independent components of $\vec{E}$ needed to completely determine the vector. The component of $\vec{E}$ in any direction s is:

$$E_s = -\partial V/\partial s.$$

For example in the radial direction (radically outward from some origin of coordinates):

$$E_r = -\frac{\partial V}{\partial r}.$$

For spherically symmetric charge distributions V depends only on r so the partial derivative with respect to r becomes the full derivative with respect to r and $\vec{E}$ has only an r-component:

$$\vec{E}(r) = E_r\hat{r} = -\frac{dV}{dr}\hat{r},$$

where $\hat{r}$ is a unit vector pointing radically outward from the center of the spherically symmetric charge distribution.

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[6] See MISN-0-147.

4. The Gradiant Operator

If you haven't been previously introduced to the gradient operation, its appearance in the relation between $\vec{E}$ and V may seem mysterious. The following may make it more plausible and acceptable.

Recall that with $\vec{E}$ given as a vector function of position in space, the force on a charge Q at any point is given by $\vec{F} = Q\vec{E}$ and the work done by this force is:

$$W = \int_A^B \vec{F} \cdot d\vec{r},$$

along some path connecting points A and B. Because $\vec{F} = Q\vec{E}$ is a conservative force, this work is independent of the path connecting these two points and the integral is just the potential energy difference between the two points:

$$W = Q\int_A^B \vec{E} \cdot d\vec{r} = E_p(A) - E_p(B) = -\Delta E_p,$$

where $\Delta E_p \equiv E_p(B) - E_p(A)$ is the change that occurs in $E_p(r)$ in going from A to B. (This may seem a trivial point, but it is an important one. You can think of many everyday examples that conform to this definition of the change in a quantity: When an airplane rises from the ground (elevation 800 ft) to its cruising altitude at 32000 ft above sea level, its change in elevation is its final elevation minus its initial elevation, or +31,200 ft. When it returns to the ground it undergoes another change in elevation, this time of -31,200 ft, again final elevation minus the initial elevation.) So we have:

$$Q\int_A^B \vec{E} \cdot d\vec{r} = E_p(A) - E_p(B)$$

$$\int_A^B \vec{E} \cdot d\vec{r} = \frac{E_p(A)}{Q} - \frac{E_p(B)}{Q} \equiv V(A) - V(B)$$

$$\int_A^B \vec{E} \cdot d\vec{r} = -\Delta V = -[V(B) - V(A)].$$

From this expression the procedure for finding the functional form of $V(\vec{r})$, given the functional form of $\vec{E}(\vec{r})$ follows by direct integration along an appropriate path.[7] If the functional form of $V(\vec{r})$ is known, how do you
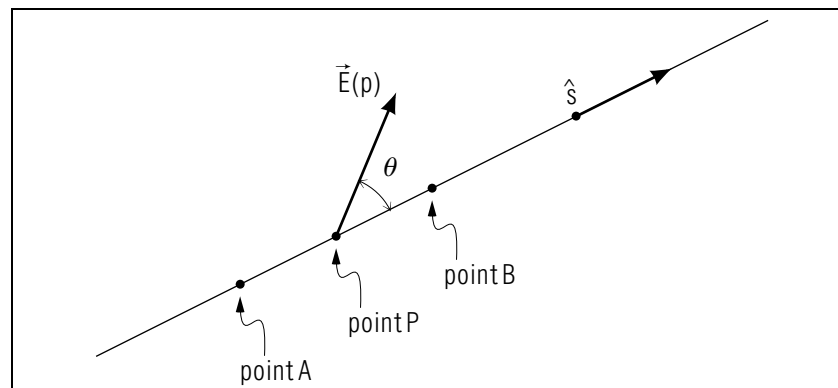
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[7]See MISN-0-147.

Figure 1. Geometric quantities involved in developing the idea of the gradient operator.

determine $\vec{E}(\vec{r})$? Again consider the left-hand side of the above equation along a specific straight line path whose direction is specified by unit vector $\hat{s}$ as in Fig. 1.

Point P is some point on the path at which the electric field has the value $\vec{E}(P)$. Here $\vec{E}$ makes an angle θ with the straight line path. Consider points A and B, also on the line, and suppose A and B are a distance ΔS apart on the line. If ΔS is small enough, $\vec{E}$ does not vary much from A to B so that we may replace $\vec{E}$ in the integral from A to B by its value at point P, $\vec{E}(P)$. (This assumes that $\vec{E}(P)$ behaves smoothly in the vicinity of P.) The component of $\vec{E}(P)$ along the line is $Es(P)$. The integral from A to B is:

$$\int_A^B E_s\, ds \approx E_s \int_A^B ds = E_s \Delta S.$$

Then:

$$-\Delta V = -[V(B) - V(A)] = \int_A^B \vec{E} \cdot d\vec{r} \approx E_s \Delta S,$$

for a path approaching an infintesimal straight line path. In the limit that points A and B shrink to point P, this becomes exactly:

$$E_s(P) = -\frac{dV}{ds}.$$

This says that the component of the vector $\vec{E}$ along the direction s at point P is the negative of the directional derivative at point P of the
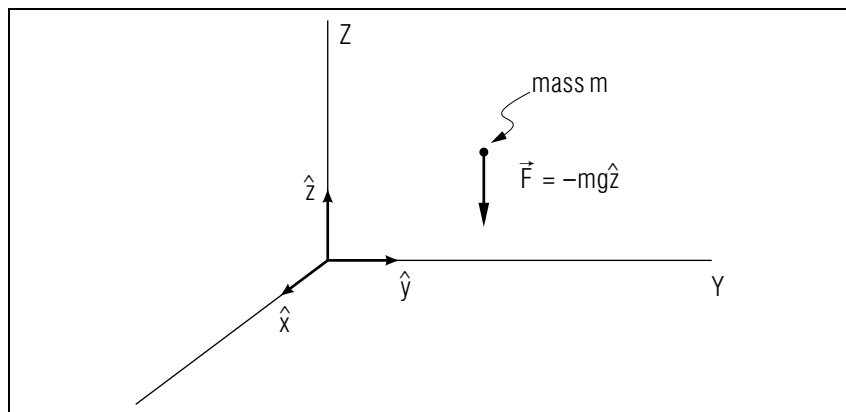
Figure 2. Quantities involved in describing gravitational force near the surface of the earth.

scalar function $V(\vec{r})$. The directional derivative is just the limit indicated above: limit as $\Delta S \to 0$ of $\Delta V/\Delta S$ where ΔV is $V(B) - V(A)$. (With $V(\vec{r})$ known at all points in the vicinity of P you can imagine that the change in V is different as you move in different directions away from point P. The directional derivative indicates the rate of change in any specified direction. If dV/ds is zero along some direction, that direction is along the equipotential line.)

So this gives you some information about $\vec{E}$ at point P: you have the component of $\vec{E}$ along the direction s. If the direction s is the x-direction, you have E_x, if s is next taken to be the y-direction, this gives you E_y, and E_z can be obtained similarly. So $\vec{E}$ is determined:

$$\vec{E}(\vec{r}) = E_x(\vec{r})\hat{x} + E_y(\vec{r})\hat{y} + E_z(\vec{r})\hat{z}$$

$$\vec{E}(\vec{r}) = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial y}\hat{y}\right).$$

$$\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r}).$$

The partial derivatives are indicated above, because if the direction s coincides with the x-direction the derivative along the x-direction is taken with y and z held constant, etc; but such a prescription is precisely what is meant by the term "partial derivative," $\partial/\partial x$. Consider as a simple and familiar example, the gravitational force near the surface of the Earth, as illustrated in Fig. 2.
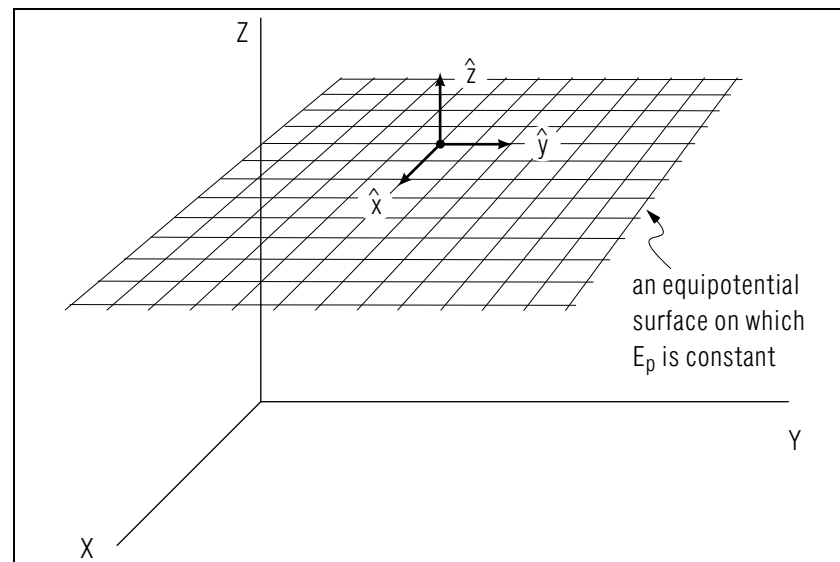
Figure 3. A gravitational equipotential surface near the surface of the earth.

The x-y plane is parallel to the surface of the Earth near the surface (very close to the surface of the Earth you may consider it a plane). The potential energy function for a mass near the surface of the Earth is:

$$E_p(x, y, z) = mgz + C,$$

where C is an arbitrary constant. The equipotential surfaces are planes, parallel to the x-y plane, on which z has a constant value (see Fig. 3).

(You can, in analogy to the electric potential, define a gravitational potential which is the potential energy per unit mass. Instead, let us continue to use the familiar potential energy.)

The force $\vec{F}$ on the mass m is given by:

$$\vec{F} = -\vec{\nabla}E_p(x, y, z)$$

$$\vec{F} = -\left(\frac{\partial E_p}{\partial x}\hat{x} + \frac{\partial E_p}{\partial y}\hat{y} + \frac{\partial E_p}{\partial z}\hat{z}\right)$$

Because E_p depends only on z, the directional derivatives in the x and y directions are zero (if you start at a point and move in those directions,

E_p doesn't change). For the z direction,

$$\vec{F} = -\frac{\partial E}{\partial z}\hat{z} = -mg\hat{z}$$

as expected.

5. Brief Answers to Assigned Problems

16.25

AF answer is OK.

16.26

Start from the result you got for the potential due to the disk of charge (answer is given in MISN-0-147) to get the electric field. The electric field at x is (using the method of Problem 16.25):

$$\vec{E} = \frac{\hat{x}Q}{2\pi\epsilon_0 R^2}\left[-\frac{x}{\sqrt{R^2 + x^2}} + 1\right].$$

Again, expanding $\sqrt{1 + R^2/x^2}$ for small R^2/x^2, this reduces to the field due to a point charge.

Acknowledgments

LOCAL GUIDE

The readings for this unit are on reserve for you in the Physics-Astronomy Library, Room 230 in the Physics-Astronomy Building. Ask for them as "The readings for CBI Unit 148." Do **not** ask for them by book title.

PROBLEM SUPPLEMENT

$$\epsilon_0 \equiv 1/(4\pi k_e) \qquad \text{and} \qquad \mu_0 \equiv 4\pi k_m$$

1. Elsewhere[8] the electric field was given and the reader was asked to calculate the potential energy and potential as a function of the distance R from the axis of a cylindrical filament of charge. The answer was:

$$E_p(R) = -\frac{\lambda Q}{2\pi\epsilon_0} \ell n\, R + C,$$

and

$$V(R) = -\frac{\lambda}{2\pi\epsilon_0} \ell n\, R + C'.$$

Starting from this result for the potential, find the electric field as a function of the distance R from the axis of the filament.

2. Inside a sphere of radius R which has total charge Q distributed uniformly throughout its volume, the electric potential at a distance r from the center of a sphere ($r < R$) is given by:
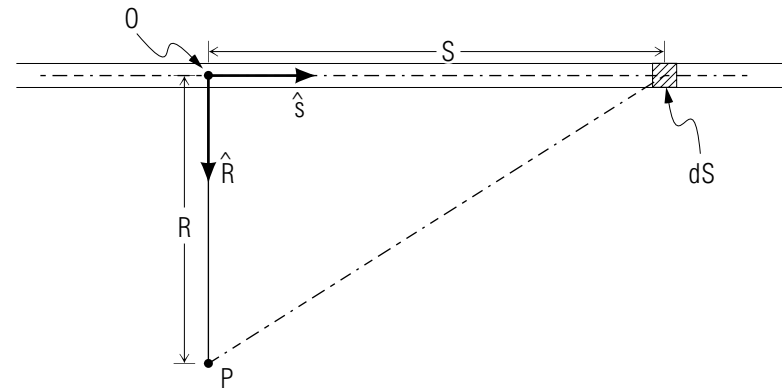
$$V(r) = -\frac{Qr^2}{8\pi\epsilon_0 R^3} + C.$$

Determine the electric field inside this sphere. Do it two ways:

   (i) first, by observing that V depends only on r so its gradient is only in the r direction; and

   (ii) second, by writing $r^2 = x^2 + y^2 + z^2$ and using the Cartesian coordinate form of the gradient.

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[8] "Electrostatic Potential Due to a Continuous Charge Distribution" (MISN-0-147), page 6.

3.

For the uniformly charged filament of Problem 1, find the electric field by direct integration. Here is guidance if you wish it. Consider the point P a distance R from the axis of the uniformly charged filament (λ coulombs per meter of length). Also consider an element of length, dS, of the filament, which is a distance S from that point on the filament (point O) which is closest to P.

a. Find the electric field at P due to the element of charge contained in the element of length dS. (Note the directions of unit vectors $\hat{R}$ and $\hat{S}$).

b. Now consider the element of charge which is also a distance S away from point O but in the opposite direction (in direction $-\hat{S}$ from O). Calculate the contribution to the electric field from this element of charge..

c. Find the resultant contribution to the field at P due to these two symmetrically placed elements of charge.

d. Sum up the contributions from all such pairs of elements located from zero to infinite distance from O. Note how symmetry has aided you in finding the total field due to this filament. Compare the method for getting this result with the method for Problem 1. Your answer should of course be the same.

4.
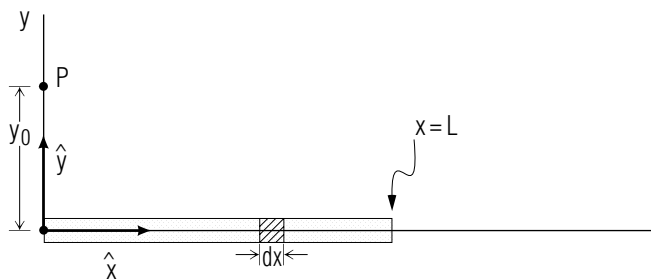
Consider now a case where you don't have the aid of symmetry you had in Problem 3 above. Consider a filament of charge uniformly distributed along the x-axis from the origin to $x = L$. The filament has a total charge Q.

a. Find the charge per unit length.

b. Consider a point P on the y-axis (located at $x = 0$, $y = y_0$). Find the contribution to the electric field at P due to an element of charge contained in element of length, dx, located at distance x from the origin along the x-axis ($x < L$).

c. Sum up the contributions due to all elements of charge from the origin to $x = L$, to find the total field at P.

d. Now take this result and let $L \to \infty$ (if you do this correctly, the answer does not get infinite!). Using this result, write down the result for the field at P due to a filament that extends from 0 to $x = -\infty$. Then add up the two contributions from two semi-infinite charge filaments to get a result identical to what you got in Problem 3.

Brief Answers:

1. Answer is in MISN-0-147.

2. Answer is in MISN-0-147 [question 2, part (a)].

3. a. $dE(P) = \dfrac{\lambda\,ds}{4\pi\epsilon_0(R^2 + S^2)^{3/2}}\left[-S\hat{S} + R\hat{R}\right].$

   b. $\dfrac{\lambda\,ds}{4\pi\epsilon_0(R^2 + S^2)^{3/2}}\left[S\hat{S} + R\hat{R}\right].$

   c. $\dfrac{\lambda R\,ds}{4\pi\epsilon_0(R^2 + S^2)^{3/2}}\hat{R}.$

   d. $E(p) = \lambda\hat{R}/(2\pi\epsilon_0)R.$

4. a. $\lambda = Q/L.$

   b. $d\vec{E}(P) = \dfrac{\lambda\,ds}{4\pi\epsilon_0(x^2 + y_0^2)^{3/2}}\left[-x\hat{x} + y_0\hat{y}\right].$

   c. $\vec{E}(P) = \dfrac{\lambda\hat{x}}{4\pi\epsilon_0}\left[\dfrac{1}{\sqrt{L^2 + y_0^2}} - \dfrac{1}{y_0}\right] + \dfrac{\lambda\hat{y}}{4\pi\epsilon_0}\left[\dfrac{L}{y_0\sqrt{L^2 + y_0^2}}\right].$

   d. From a line of charge extending from 0 to $L = \infty$:

     $\vec{E}(P, L \to \infty) = -\dfrac{\lambda\hat{x}}{4\pi\epsilon_0 y_0} + \dfrac{\lambda\hat{y}}{4\pi\epsilon_0 y_0}.$

     From a line of charge extending from 0 to $L = -\infty$:

     $\vec{E}(P, L \to -\infty) = \dfrac{\lambda\hat{x}}{4\pi\epsilon_0 y_0} + \dfrac{\lambda\hat{y}}{4\pi\epsilon_0 y_0}.$

     From the sum of the above two lines of charge:

     $\vec{E}(P) = \dfrac{\lambda\hat{y}}{2\pi\epsilon_0 y_0}.$

MODEL EXAM

1. See Output Skills K1-K3 in this module's *ID Sheet*. One or more of these skills may be on the actual exam.

2. A uniformly distributed line of charge is placed along the x-axis from $x = 0$ to $x = a$. There is a total charge Q distributed along this line.

   a. Write down the expression for the electric field at point P along the x-axis a distance x from the origin due to an infinitesimal segment of charge contained in an infinitesimal segment of length ds at a distance s from the origin. [C]

   b. Summing up the contributions from all of the elements of charge, find the expression for the electric field at P due to this line of charge. [E]

   c. The electric potential at P due to this line of charge is given by:

   $$V(x) = \frac{Q}{4\pi\epsilon_0 a} \ell n \left(\frac{x}{x - a} \right).$$

   d. From this potential function (for points along the x-axis), find the function that gives the electric field at points along the x-axis. [G]

2. a. Write down the function that gives the potential energy at a distance r from the origin when there is a point charge Q at the origin. [B]

   b. Evaluate the gradient of this potential function to determine the electric field at distance r from the origin. [F]

3. For a parallel plate capacitor, the potential increases linearly from one plate to the other if the distance between the plates is small compared to the size of the plates. Assume this to be true for a capacitor where the potential increases from V_1 on plate #1 to V_2 on plate #2, a distance D away. Assume the plates lie in the x-z plane and they are separated by a distance D. Use the gradient operator to determine the electric field between the plates. [D]

Brief Answers:

A. See Exercises, MISN-0-147.

B. $\dfrac{Q}{4\pi\epsilon_0 r}$

C. $\dfrac{\lambda \, ds}{4\pi\epsilon_0 (x - s)^2} \, \hat{x}$.

D. $\vec{E} = -\dfrac{\partial V}{\partial y} \, \hat{y}$; $\dfrac{\partial V}{\partial y} =$ slope of curve $= \dfrac{V_2 - V_1}{D}$,

   so $\vec{E} = -\dfrac{V_2 - V_1}{D} \, \hat{y}$, points from plate 2 to plate 1.

E. $\vec{E}(x) = \dfrac{\lambda}{4\pi\epsilon_0} \left[\dfrac{1}{x - a} - \dfrac{1}{x} \right] \hat{x}$; $\qquad (x > a)$.

F. $\vec{\nabla} V = -\dfrac{dV}{dr} \, \hat{r} = \dfrac{Q}{4\pi\epsilon_0 r^2} \, \hat{r}$.

G. $\vec{\nabla} V = -\dfrac{\partial V}{\partial x} \, \hat{x}$, gives the same $\vec{E}$ as in answer (E) above.