PHYSNET

MISN-0-147
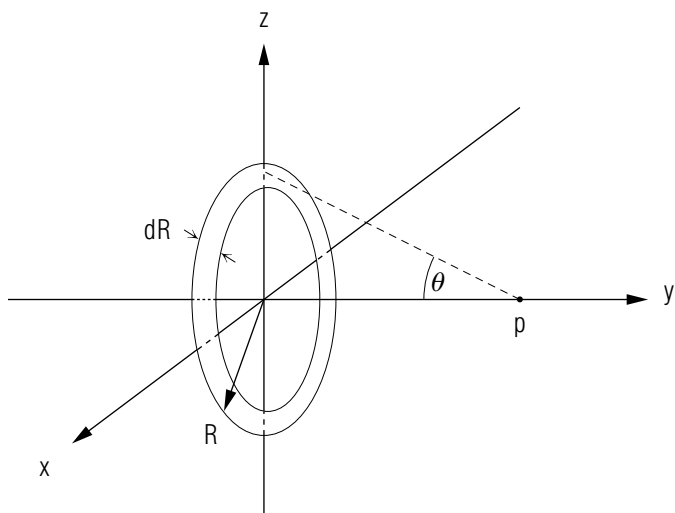
ELECTROSTATIC POTENTIAL
FROM DISTRIBUTED CHARGES

ELECTROSTATIC POTENTIAL FROM DISTRIBUTED CHARGES

by

J. Kovacs and P. Signell
Michigan State University

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Title: **Electrostatic Potential from Distributed Charges**

Author: J. Kovacs and P. Signell, Dept. of Physics, Mich. State Univ.

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Input Skills:

1. Express the electrostatic potential due to a point charge (MISN-0-116).
2. Expand binomials, like $(a + x)^{1/2}$, using a Taylor series (MISN-0-4).
3. Integrate simple functions, such as $(x^2 + a^2)^{1/2}$, between definite symbolic or numerical limits (MISN-0-1).

Output Skills (Knowledge):

K1. Vocabulary: continuous charge distribution, linear charge density, surface charge density, uniform charge distribution, volume charge density.

K2. State the general expression for the potential at a point in space due to a continuous charge distribution.

Output Skills (Problem Solving):

S1. Given a linear charge distribution in the shape of a straight line, find the electric potential at any point, on the line, that is outside the charge distribution.

S2. Given a circular arc or disc of charge, find the electric potential at any point on the rotational axis of symmetry.

Post-Options:

1. "The Electrostatic Field Due to Continuous Charge Distributions" (MISN-0-148).
2. "Electric Fields from Symmetric Charge Distributions" (MISN-0-153).

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1. Introduction

The electric potential at a point in space is defined as the potential energy per unit charge of a test charge placed at that point. Alternatively potential may be defined as the work per unit charge that must be done by an external agent to bring a charged particle from infinity to that point. From these definitions, the potential associated with any electric field may be determined. Starting from the expressions for the electric potential due to a point charge and due to a collection of point charges, a general procedure can be developed whereby the potential due to a continuous distribution of charge can be obtained if the spatial distribution of the charge is known. We can use this general procedure to obtain the potential due to, for example, a continuous line of charge, a uniform circular distribution of charge, or a uniform sheet of charge. The potential from spherical distributions of charge can also be obtained by such methods but will not be treated here.

2. Discrete Charges

2a. A Single Discrete Charge. The electrostatic potential at a point P in space, due to a charge Q_1 on a point particle that is a distance r_{1P} from P, is given by:

$$V(P) = k_e \frac{Q_1}{r_{1P}} \tag{1}$$

The zero point, the point where the potential due to this charge Q_1 is zero, has been taken to be at an infinite distance from Q_1, i.e., at $r_{1P} = \infty$. This is a good choice for the zero point since we generally expect quantities to die away as we get far from their sources.[1] The value of this potential will naturally be negative if Q_1 is negative and positive if Q_1 is positive. Because this is the potential due to a point-particle charge, it belongs to the class of potentials called "Coulomb Potentials".

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[1] See "Potential Due To Discrete Charges" (MISN-0-116).

2b. A Set of Discrete Charges. Consider the effect on the value of the potential at P, due to a single nearby charge Q_1, if a second point charge Q_2 is placed somewhere in the vicinity of P, a distance r_{2P} from P. Its contribution is given by an expression similar to Eq. (1), except that the charge is Q_2 and its distance from point P is r_{2P}. However, if Q_1 remains where it was, the presence of Q_2 does not alter the contribution of Q_1 to the potential at P. Each charge contributes independently to the potential at P and the electric potential there is simply the sum of the two contributions.[2] so the net potential at P due to the two charges is

$$V(P) = V_1(P) + V_2(P) = k_e \left(\frac{Q_1}{r_{1P}} + \frac{Q_2}{r_{2P}} \right) \tag{2}$$

This expression can be generalized to a collection of charges, each of which contributes to the potential at the point P, depending upon its charge Q_i and its distance r_{iP} from P. If there are N discrete charges contributing to the potential at P, then

$$V(P) = k_e \sum_{i=1}^{N} \frac{Q_i}{r_{iP}} , \tag{3}$$

where the index i, running from 1 to N, serves to label the charges [just as the subscripts 1 and 2 do in Eq. (2)].

3. Linear Distributions

3a. Discrete Charges on a Circle. As an illustration, consider 400 particles, each having charge q, arranged randomly around the circumference of a circle of radius R. What electrostatic potential do these charges create at the center of the circle? Note that: (1) each charge is the same distance R from the point at the center of the circle, independent of where the charge is on the circle; and (2) all particles have the same charge q. This means that each particle produces an identical contribution to the potential at the center of the circle.[3] The contribution of the charge on

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[2] The potentials from the individual charges are said to "add as scalars." However, the electric field contributions at the point P due to several charges are said to "add as vectors," meaning that the electric field is the vector sum of the fields that each of the charges would produce at P if it alone was present.

[3] Note that the *electric field* at the center of the circle *does* depend on the arrangement of the charges. This is because electric field contributions add as vectors, in contrast to the potential contributions which add as scalars.

the nth particle can be written:

$$V_n(0) = k_e \frac{q}{R}, \tag{4}$$

Then the total potential at the center of the ring is just the number of charges times the potential from any one of the charges:

$$V(0) = 400 \, k_e \frac{q}{R}, \tag{5}$$

and this is just the same potential as would be produced by a single particle having charge $Q = 400 \, q$ located a distance R away from the center of the circle in any direction:

$$V(0) = k_e \frac{Q}{R}. \tag{6}$$

Thus for the special case of a circular arrangement of equal charges, we can easily calculate the potential at the center of the circle. On the other hand, if there are N particles on the circle but they are not equally charged, then:

$$V(0) = \sum_{n=1}^{N} k_e \frac{q_n}{R} = \frac{k_e}{R} \sum_{n=1}^{N} q_n = k_e \frac{Q}{R}, \tag{7}$$

where Q is the total charge on the circle. Note the similarity to the Coulomb potential, Eq. (1).

3b. Continuous Charges on a Circle. The case of discrete charges distributed around a circle can be extended to the situation where there are so many charges, closely packed on the circle, that we can more easily describe the charge in terms of a "linear charge density," the amount of charge per unit length along the line that is the circumference of the circle (this linear charge density is often given in coulombs/meter). In general, the charge density will vary from point to point around the circle. Each point on the circle will be labeled by its distance around the circle from some specified starting point such as, say, the right-most point on the circle. Thus, for example, we might say that a particular point is $\ell = 2.3$ cm counterclockwise around the circle from the circle's right-most point, and that the linear charge density there 1.4×10^{-5} coulombs/meter. If the charge density is the same at all points around the circle we say that the charge is "uniformly distributed" along that circular line.

3c. Linear Charge Density at a Point. To define the linear charge density at any one point mathematically, we start with a small region around that point. We will call the point in question ℓ_P, where ℓ is distance measured along the line from some designated starting point. We focus our attention on a small segment of the line that includes the point ℓ_P. We denote the length of the segment of line by the symbols $\Delta \ell$. We denote the actual amount of charge in that segment by the symbols ΔQ. We now define the linear charge density λ at the point P as the limit of the ratio of ΔQ to $\Delta \ell$ as we shrink $\Delta \ell$ to zero around the point P.[4] This is just the definition of the derivative of the charge with respect to distance along the line at the point P:

$$\lambda(\ell_P) = \lim_{\Delta \ell \to 0} \frac{\Delta Q}{\Delta \ell} = \frac{dQ}{d\ell}. \tag{8}$$

To find the amount of charge in some finite-length region R, we integrate over that region:

$$Q_R = \int_R dQ = \int_R \left(\frac{dQ}{d\ell} \right) d\ell = \int_R \lambda \, d\ell. \tag{9}$$

If the charge is uniformly distributed over a length L along a line of charge, the total charge on the line, Q_L, is just the (uniform) charge density, λ, times the line-length, L:[5]

$$Q = \lambda L \qquad \text{(uniform } \lambda\text{)}. \tag{10}$$

$\triangleright$ Show that a circle of radius 5.0 cm having a uniformly-distributed total charge 2.0×10^{-6} C has a charge density of 6.4×10^{-6} C/m. If charge is distributed non-uniformly around a circle, the value of the linear charge density λ varies from point to point around the circle. If distance around the circle from some designated starting point is called ℓ, then we can indicate the variation of λ by writing it as a function of position, $\lambda(\ell)$.

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[4]The idea of a "limit" begins with the numerical value for the ratio for a particular segment of line around P. If the length of the segment is small enough, then making the even smaller will make an insignificant difference in the ratio. For example, cutting the length in half might make a difference in the seventh significant digit of the ratio. Cutting it in half again might provide a further change in only the eighth significant digit of the ratio. As the length shrinks toward zero the changes in the ratio shrink toward zero, resulting in a definite single value for the ratio at the point.

[5]Notice in Eq. (3) that a constant λ can be taken outside the integration sign and the integration then just produces the total length, L.
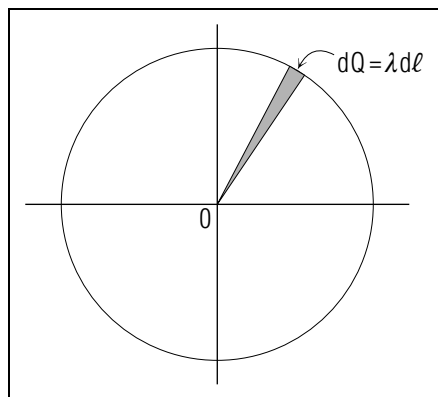
Figure 1. An infinitesimal amount of charge dQ is related to the infinitesimal length $d\ell$ in which it occurs through the linear charge density λ.

3d. Uniform Distribution on a Circle. If the charge is uniformly distributed around a circle, the linear charge density around the circle is:

$$\lambda = \frac{Q}{2\pi R}\,; \qquad \text{(uniform distribution).} \qquad (11)$$

The resulting potential at the center of the circle, in terms of r and λ, is as in Eq. (6):

$$V(0) = k_e 2\pi\lambda\,; \qquad \text{(uniform distribution).} \qquad (12)$$

▷ Note that the answer is independent of the radius of the circle. Why?

3e. Non-Uniform Distribution on a Circle. To compute the potential at the center of a circle of non-uniformly distributed charges, we first rewrite Eq. (8) to show the infinitesimal (see *Help: [S-5]*) amount of potential, dV, produced at some space-point, $\vec{r}$, due an infimitesimal amount of charge dQ at space-point $\vec{r}\prime$ (see Fig. 1):

$$dQ = \lambda d\ell\,.$$

$$dV = k_e \frac{dQ}{|\vec{r}\prime - \vec{r}|}$$

$$V(\vec{r}) = \int k_e \frac{dQ}{|\vec{r}\prime - \vec{r}|} = \frac{k_e}{|\vec{r}\prime - \vec{r}|} \int \lambda d\ell$$

$$V(\vec{r}) = \frac{k_e}{|\vec{r}\prime - \vec{r}|} Q$$

which has the same form as the Coulomb potential in Eq. (1) (although the electric filed will not be the same as in the Coulomb case).

4. Surface Distributions

4a. Surface Charge Density. Suppose we wish to determine the electrostatic potential at some point near a surface which has charged particles distributed over it in such a fashion that the charge distribution can be considered to be continuous.[6] Then we can define the density of charges on the surface much as we define mass density in working with material objects. A typical unit of measure for charge density is "coulombs per square meter." To define the surface-charge density at any point mathematically, we must consider a small region around the point. We will call the point in question P and the small region's area ΔA. The size and shape of the region will soon be seen to be unimportant since we will mathematically change it. We represent the charge in the region by ΔQ. We now define the surface charge density σ as the limit of the ratio of region's charge to area as the region shrinks toward zero area around the point P. This is just the definition of the derivative of the charge with respect to area at the point P:[7]

$$\sigma(P) = \lim_{\Delta A \to 0} \frac{\Delta Q}{\Delta A} = \frac{dQ}{dA}\,.$$

To find the amount of charge in some region R, we integrate over that region:

$$Q = \int_R dQ = \int_R \left(\frac{dQ}{dA}\right) dA = \int_R \sigma\, dA\,. \qquad (13)$$

If charge is said to be uniformly distributed over some surface, it means that σ is the same at all points on the surface and the total charge on the surface is then just the (uniform) charge density times the area:[8]

$$Q = \sigma A \qquad \text{(uniform } \sigma\text{).} \qquad (14)$$

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[6]This means that the particles are so close together that they appear to be continuous, much as your skin appears to be continuous even though it is made up of individual atoms. Mathematically, we treat charges as continuous when doing so will introduce an error so small that it will have no significant effect on answers.

[7]The idea of a "limit" begins with the numerical value for the ratio for a particular area around P. If the area is small enough, making the area even smaller will make an insignificant difference in the ratio. For example, halving the area might make a difference in the seventh significant digit in the ratio. Halving it again might provide a further change in only the eighth significant digit in the ratio. As the area shrinks from this region toward zero the changes in the ratio shrink toward zero, resulting in a definite single value.

[8]Notice in Eq. (3) that a constant σ can be taken outside the integration sign and the integration then just produces the total area, A.
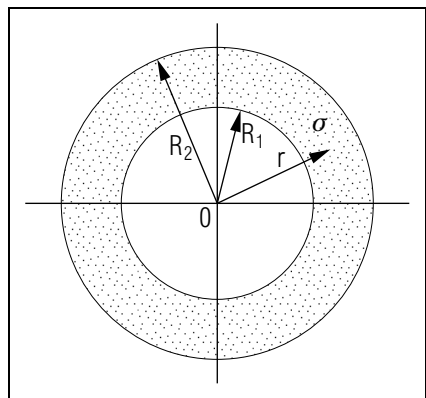
Figure 2. A hollow disk has charge density σ for $R_1 < r < R_2$.

▷ Show that a (circular) disk of radius 5.0 cm having a uniformly-distributed total charge 2.0×10^{-6} C has a charge density of 2.5×10^{-4} C/m^2.

4b. Continuous Charge on a Disk. Suppose we have a hollow disk, a flat ring, as in Fig. 2, with uniform charge density σ, and we want to calculate the potential at the center of the ring. Using the usual technique for such cases, we first calculate the infinitesimal potential dV due to a sub-ring of width dr at radius r as in Fig. 3. (*Help: [S-5]*) and then integrate over the area defined by $R_1 < r < R_2$:

$$V = \int_{r=R_1}^{r=R_2} dV \, . \tag{15}$$
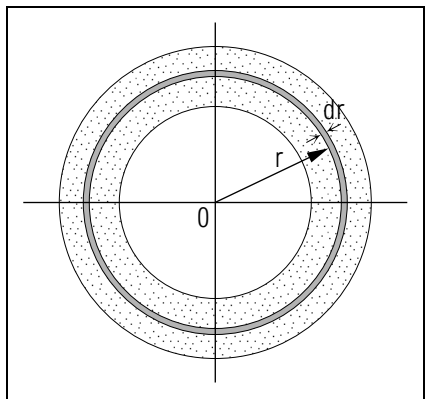
Figure 3. A circular ring with infinitesimal width dr has an infinitesimal area $dA = 2\pi r dr$. This is the same disk as in Figref1.

To get the potential dV at the center due to a ring of charge having radial width dr, we need the charge dQ in that ring and the distance of the ring from the center. The charge is the area times the density, $2\pi r \, dr$ times σ, so we get:

$$dV = k_e \frac{dQ}{r} = k_e \frac{\sigma \, 2\pi r \, dr}{r} \, , \tag{16}$$

▷ If you do not see how Eq. (16) is arrived at, be sure to go through the assistance provided in this module's Special Assistance Supplement. *Help: [S-5]*

Putting Eq. (16) into Eq. (15) gives us the potential:

$$V(0) = \int_{R_1}^{R_2} k_e \frac{\sigma \, 2\pi r \, dr}{r} = k_e \sigma \, 2\pi \int_{R_1}^{R_2} dr = k_e \sigma \, 2\pi (R_2 - R_1) \, .$$

The total charge Q on the ring is just the density times the area:

$$Q = \sigma \, (\pi R_2^2 - \pi R_1^2) \, ,$$

so the answer in terms of the total charge is:

$$V(0) = \frac{2k_e Q}{R_2 + R_1} = \frac{k_e Q}{R_{av}} \, , \tag{17}$$

where R_{av} is the simple average of the two radii. Note the similarity to a Coulomb potential. In fact, if the width of the ring is very small compared to R_1, all distances from charge to the center of the ring will be very close to the single value R_{ave}. Calling that single value R, Eq. (17) becomes the circular line answer of Eq. (7). Put into words, a ring whose width is very small compared to any of its radii will "look" from the center point as if it is very close to being a circular line of charge and so the potential at the center will be very close to the answer for a circular line of charge.

5. Continuous Distributions In General

5a. General Approach. In general, we say that an infinitesimal part of the charge distribution, dQ, produces an infinitesimal Coulomb-like contribution[9] $dV(\vec{r})$ to the total potential at a point $\vec{r}$:

$$dV(\vec{r}) = k_e \frac{dQ(\vec{r}\prime)}{|\vec{r}\prime - \vec{r}|} \, , \tag{18}$$

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[9] It is Coulomb-like because the charge region is so small that it resembles a point particle; more on this later.

where $|\vec{r}\prime - \vec{r}|$ is the distance from the infinitesimal piece of charge at point $\vec{r}\prime$ to the point $\vec{r}$ at which we wish to know the potential. Then the equation for the potential itself is obtained by integrating both sides of Eq. (18):

$$V(\vec{r}) = k_e \int \frac{dQ(r\prime)}{|\vec{r}\prime - \vec{r}|} \,. \tag{19}$$

The integral is easily performed for certain geometrical configurations of charge, but for other configurations computer integration must be used.

5b. Some Specific Geometries. We interpret the quantity $dQ(\vec{r})$ on the right side of Eq. (19) as the charge in some infinitesimally small region of space. Then we use an appropriate density function to convert the integral from one over charge to one over space. The three generally-used density functions are:

$$\lambda = \frac{dQ}{d\ell}; \qquad \sigma = \frac{dQ}{dA}; \qquad \rho = \frac{dQ}{d\mathcal{V}} \,.$$

where:

$\lambda(\ell) \equiv$ "linear charge density"

$\equiv$ "charge per unit length along a one-dimensional line,"

$\sigma(\vec{r}) \equiv$ "surface charge density"

$\equiv$ "charge per unit area on a two-dimensional surface,"

$\rho(\vec{r}) \equiv$ "volume charge density"

$\equiv$ "charge per unit volume in three-dimensional space."

Note that λ has a value at each point along a line of charge, with any specific point being designated by some one-dimensional position descriptor. Similarly, σ has a value at each point on a surface of charge, with any specific point being designated by some two-dimensional position descriptor. Finally, ρ has a value at each point throughout a three-dimensional distribution of charge, with any specific point being designated by some three-dimensional position descriptor.

5c. Spatial Representations of the Charge Elements. In order to integrate over space in any particular problem, one must first use the appropriate density function to convert from infinitesimal elements of charge to infinitesimal elements of space. The exact method of doing this depends on the spatial symmetries of the problem, if any exist, in order to ease the integration process.

For a linear charge distribution we may write:

$$dQ = \lambda(\ell) \, d\ell \,,$$

and then we can integrate along the line of charge from one end to the other. Any line of charge is really three-dimensional but we can treat it as one-dimensional if the cross-sectional area is small compared to the distances from all parts of the charged line to the point where we wish to calculate the potential (so it is far enough away so it "looks" like a line of charge).

For surface charge distributions we write:

$$dQ = \sigma(x, y) \, dx \, dy \,,$$

or:

$$dQ = \sigma(r, \theta) \, r \, dr \, d\theta \,,$$

depending on whether the charge density is described in Cartesian or circular coordinates. In other words, the infinitesimal element of area is $dA = dxdy$ in Cartesian coordinates, etc. Any surface of charge is really three-dimensional but we can treat it as two-dimensional if its thickness is small compared to the distances from all parts of the charge to the point where we wish to calculate the potential (so it is far enough away so it "looks" like a surface of charge).

For volume charge distributions we write:

$$dQ = \rho(x, y, z) \, dx \, dy \, dz \,,$$

or:

$$dQ = \rho(r, \theta, \phi) \, r^2 \, \sin\theta \, d\theta \, d\phi \, dr \,,$$

or:

$$dQ = \rho(r, \phi, z) \, r \, dr \, d\phi \, dz \,,$$

depending on whether the charge density is described in Cartesian, spherical or cylindrical coordinates (other coordinate systems may also be used). In other words, the infinitesimal element of volume is $d\mathcal{V} = dxdydz$ in Cartesian coordinates, etc.

Acknowledgments

Glossary

- **continuous charge distribution**: an arrangement of many discrete charges so closely spaced that the charge is treated as a continuum, resulting in a replacement of discrete sums with integrals. Such charge distributions occur over one dimension (lines), two dimensions (surfaces) or three dimensions (volumes).

- **linear charge density**: on a linear charge distribution, the ratio of an infinitesimal portion of charge to the infinitesimal element of length it is contained in. This ratio varies over the length of charge distribution unless the charge is uniformly distributed. For a uniform distribution, the linear charge density, λ, is just the total charge divided by the total length of the distribution.

- **surface charge density**: within a surface charge distribution, the ratio of an infinitesimal portion of charge to the infinitesimal element of surface area it is contained in. This ratio varies over the surface of the charge distribution unless the charge is uniformly distributed. For a uniform distribution, the surface charge density, σ, is just the total charge divided by the total area of the distribution.

- **uniform charge distribution**: any distribution of charge that is independent of position within the distributed charge.

- **volume charge density**: within a volume charge distribution, the ratio of an infinitesimal portion of charge to the infinitesimal element of volume it is contained. This ratio varies over the volume of the charge distribution unless the charge is uniformly distributed. For a uniform distribution, the volume charge density, ρ, at any point is just the total charge divided by the total volume of the charge distribution.

   Treating the charges as though they are continuous rather than discrete will introduce an error so small it will be unobservable in almost any application, and it will allow us to easily solve problems that could otherwise be attempted only with horrendous amounts of effort and computer time. When charges are distributed along a line, we can designate points along the line by the distance ℓ measured along the line from some designated zero point, a point where $\ell = 0$, so the amount of charge per unit length along the line, at any point ℓ, can be written as $\lambda(\ell)$. We generally say that we can represent the

PROBLEM SUPPLEMENT

Note: Problem 5 also occurs in this module's *Model Exam*.

1.
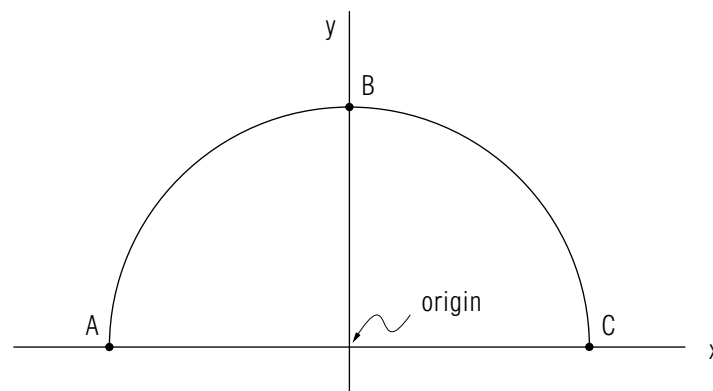
   a. Determine the potential at the origin of coordinates due to the semicircular line of charge shown in the above figure. The center of the arc is at the origin, its radius is 0.04 m. A total charge of 1.0×10^{-6} C is distributed uniformly along this semicircular line.

   b. Suppose this 10^{-6} C is distributed differently, with 7.5×10^{-7} C distributed uniformly along the quarter circular arc AB and 2.5×10^{-7} C distributed uniformly along the quarter circular arc BC. Now what is the potential at the origin?

2. Consider a point P which is on the line through the center of a circle of radius R and perpendicular to the plane of that circle. P is located at a distance x from the center of the circle. If a charge per unit length λ is distributed uniformly on the circumference of that circle, what is the electrostatic potential at P? Check that the result obtained is consistent with the results of Eq. (1) by letting x go to zero so that point P is at the center of the circle.

3. Consider a circular distribution of charge similar to the circular line of charge in Problem 2, except that the charge is distributed on a flat circular ring of infinitesimal thickness, whose inside radius is R and outside radius is $R + dR$ as shown in the sketch below.
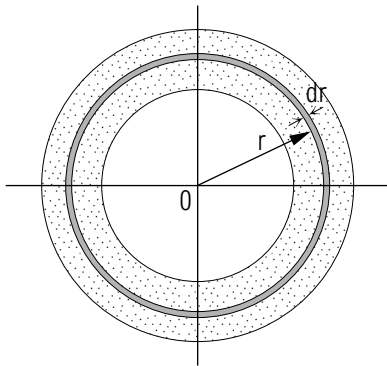
a. If charge is uniformly distributed on the shaded portion of the above ring with a surface density σ what is the total charge on the ring?

b. Consider a line perpendicular to the plane of the ring and passing through its center O. At a point P on this line, a distance x above the plane of the ring, determine the electrostatic potential due to the charge on the ring.

4. a. Suppose charge is distributed uniformly on a circular disk of radius R. Treating this disk as divided into rings, calculate the potential at a point P on a line passing through the center of the disk. The line is perpendicular to the plane of the disk and the point P is a distance x along the line from the center of the disk. You can make use of the result of Problem 3b. Express your answer in terms of the total charge Q on the disk. *Help: [S-2]*

b. (only for those interested) For the case $x \gg R$, the point P is so far from the disk that as far as the potential at point P is concerned, the disk looks like a point charge. In that case the answer to part (a) should reduce to the potential due to a point charge at a distance x from the charge. Assume $x \gg R$ and expand the $(R^2 + x^2)^{1/2}$ term using a Taylor series expansion,[10] and show that you do indeed get the potential from a point charge. *Help: [S-1]*

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[10]See "Taylor's Polynomial Approximation For Functions," (MISN-0-4).
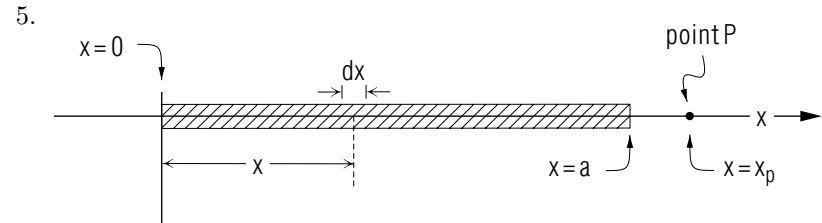
5.

A uniformly distributed line of charge is placed along the x-axis from $x = 0$ to $x = a$. The total charge on the line is Q.

a. Write down the expression for the potential, at a point P on the x-axis a distance x_P from the origin, due to an infinitesimal segment of charge contained in an infinitesimal segment of length dx at a distance x from the origin (see the sketch).

b. Summing up the contributions from all of the elements of charge, find the expression for the potential at P due to this line of charge. Assume $x_P > a$.

c. (only for those interested) For the case $(x_P \gg a)$, show that $V(x_P) \approx k_e Q / x_P$, meaning that when the charged line is *far enough away* so its size appears to be negligible, it looks like a point charge. *Help: [S-8]*

Brief Answers:

1. a. 2.25×10^5 V

   b. 2.25×10^5 V

2. $V(P) = k_e 2\pi \dfrac{\lambda R}{(R^2 + x^2)^{1/2}}$

3. a. $dQ = 2\pi \sigma R \, dR$ *Help: [S-3]*

   b. $dV(P) = k_e 2\pi \dfrac{\sigma R \, dR}{(R^2 + x^2)^{1/2}}$

4. a. $V(x) = k_e 2 \dfrac{Q}{R^2} \left[(R^2 + x^2)^{1/2} - x \right]$

   b. $V(x) = k_e 2 \dfrac{Q}{R^2} \left[(R^2 + x^2)^{1/2} - x - R \right]$

5. a. $dV(x_P) = k_e \dfrac{Q dx}{a(x_P - x)}$

   b. $V(x_P) = k_e \dfrac{Q}{a} \ell n \left(\dfrac{x_P}{x_P - a} \right)$ *Help: [S-4]*

SPECIAL ASSISTANCE SUPPLEMENT

S-1   *(from PS, Problem 4b)*

With $x \gg R$, the ratio R/x obeys the relation: $R/x \ll 1$. When that is true, it is even truer that: $(R/x)^2 \ll 1$. Then we can rewrite the expression for the potential in terms of the small quantity $\epsilon \equiv R/x$ and use a particular small-quantity approximation (here written as a function of some variable y):[a]

$$(1 + y)^{1/2} \approx 1 + \frac{y}{2}. \qquad (y \ll 1)$$

Here is the potential, $V(x)$, rewritten in terms of ϵ:

$$V(\epsilon) = k_e \frac{2Q}{R^2} \left[\left(R^2 + \frac{R^2}{\epsilon^2} \right)^{1/2} - \frac{R}{\epsilon} \right],$$

and here it is again, with R/ϵ factored out:

$$V(\epsilon) = k_e \frac{2Q}{\epsilon R} [(1 + \epsilon^2)^{1/2} - 1],$$

and here it is after the small-ϵ^2 approximation has been applied:

$$V(\epsilon) \approx k_e \frac{2Q}{\epsilon R} \left[\frac{\epsilon^2}{2} \right]. \qquad (\epsilon \ll 1)$$

Finally, here it is converted back to a function of x:

$$V(x) \approx k_e \frac{Q}{x}, \qquad (x \gg R)$$

which is truly Coulombic.

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[a]See "Taylor's Polynomial Approximation For Functions," (MISN-0-4).

S-2   *(from PS, Problem 4a)*

You are going to have to integrate on R to sum the contributions of the various rings. However, the symbol R has already been defined in this problem as the radius of the *disc* so it cannot also be used as the symbol for the (variable) radius of the *ring*. We switch the ring radius to the symbol R' and that is what becomes the variable of integration as we sum on the various rings that make up the disc. You can look up the value of the integral in a Table of Integrals, or you can do it by hand this way: substitute $y^2 \equiv (R'^2 + x^2)$ so the denominator is y and the numerator is $y\,dy$, with the limits of integration being x and $\sqrt{R^2 + x^2}$.

S-3   *(from PS, Problem 3a)*

The notation "dR" and "dQ" indicates that these are "infinitesimal quantities," quantities that will be made vanishingly small when they are used (as they must be) in integrals or derivatives. Only when finite quantities are made from them can they be compared to real-world measured quantities.

Considering them to be infinitesimals, the circumference *at any radius within that infinitesimal increment of radius* is just $2\pi R$ and so the (infinitesimal) area of the ring is $2\pi R\,dR$ and the (infinitesimal) charge within the ring is $\sigma\,2\,\pi\,R\,dR$.

S-4   *(from PS, Problem 5b)*

from module 1's cover:
$$\int \frac{dx}{x} = \ell n\,x$$
.

S-5   *(from TX, Sect. 4b)*

The "infinitesimals" method generally used in science and technology to find the integrand of an unknown integral departs somewhat from the mathematically rigorous limit technique. Nevertheless, the "infinitesimals" method requires an understanding of the mathematical limit technique for its successful use. A good test bed for understanding is the use of each method to derive the integrand for determining the area of a (circular) disk, where you can perform the integral and make sure you get the known answer of πr^2.

▷ First, make sure you can derive the integrand using the limit method. *Help: [S-6]*

▷ Then use your understanding of the limit method to derive the same integrand using the usual science/technology method. *Help: [S-7]*

S-6   *(from AS, [S-5])*

In the "mathematical limit" method we start by writing the area as:

$$A = \int_{r=0}^{r=R} dA = \int_0^R \frac{dA}{dr} \, dr \,. \tag{1}$$

Now we must find the integrand, dA/dr. We calculate the derivative by considering a part of the disk, a circular ring of width Δr that is between r and $r + \Delta r$. From this ring we get the dericative by taking the limit of the finite-element ratio as the width of the ring shrinks toward zero:

$$\frac{dA}{dr} = \lim_{\Delta r \to 0} \frac{\Delta A}{\Delta r} \,.$$

Now:

$$\Delta A = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r \Delta r + \pi(\Delta r)^2 \,. \tag{2}$$

Then:

$$\frac{dA}{dr} = \lim_{\Delta r \to 0} \frac{\Delta A}{\Delta r} = \lim_{\Delta r \to 0} (2\pi r + \pi \Delta r) = 2\pi r \,. \tag{3}$$

Notice that the $(\Delta r)^2$ term of Eq. (2) did not survive into the final answer for the integrand. Then we have, finally:

$$A = \int_{r=0}^{r=R} dA = \int_0^R \frac{dA}{dr} \, dr = \int_0^R 2\pi r \, dr = \pi R^2 \,. \tag{4}$$

Of course we could have differentiated this answer to obtain the integrand but we wanted to illustrate the method so we can use it in situations where we do not know the answer beforehand.
▷ Now use your understanding of the limit method to derive the same integrand using the usual science/technology method. *Help: [S-7]*

S-7   *(from AS, [S-5])*

The method usually used in science and technology to construct an integrand *reinterprets* the statement of the variable of integration, "dA" in Eq. (1) of this supplement, as an actual "infinitesimal" area whose value is a length multiplied by a width. In our case the length is the finite circumference of the ring (the single value for the length is discussed several sentences further on), $2\pi r$, while the width has the "infinitesimal" value dr. Writing the (infinitesimal) area dA as the product of that length times that width gives us:

$$dA = 2\pi r \, dr$$

so the integral is:

$$A = \int_{r=0}^{r=R} dA = \int_0^R 2\pi r \, dr = \pi R^2 \,.$$

The integrand derived here agrees exactly with the rigorously derived integrand in Eqs. (2)-(4) of this supplement. Note that since the width dr is "infinitesimal," the circumference can be considered to be the same at r and $r + dr$. Of course the real reason those two circumferences can be considered to be the same is that the difference disappears if we take a proper limit as in Eq. (3) of this supplement.

S-8   *(from PS, problem 5c)*

With $x_P \gg a$ we have $a/x_P \ll 1$, so we can rewrite the expression for the potential in terms of the small quantity $\epsilon \equiv a/x_P$ and use these two small-quantity approximations:[a]

$$\frac{1}{1-\epsilon} \approx 1 + \epsilon, \qquad (\epsilon \ll 1)$$

and

$$\ell n \,(1 + \epsilon) \approx \epsilon. \qquad (\epsilon \ll 1)$$

Here is the equation rewritten in terms of ϵ:

$$V(\epsilon) = k_e \frac{Q}{a} \, \ell n \left(\frac{1}{1 - \epsilon} \right)$$

and here it is after the two small-quantity approximations have been applied and the answer has been restored to a function of x_P:

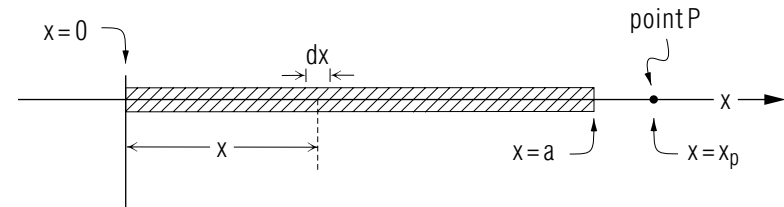$$V(x_P) \approx k_e \frac{Q}{x_P} \qquad (a \ll x_P).$$

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[a]See "Taylor's Polynomial Approximation For Functions," (MISN-0-4).

MODEL EXAM

$$\int \frac{x \, dx}{\sqrt{x^2 + a^2}} = \sqrt{x^2 + a^2}$$

1. See Output Skills K1-K2 in this module's *ID Sheet*.

2.

A uniformly distributed line of charge is placed along the x-axis from $x = 0$ to $x = a$. The total charge on the line is Q.

a. Write down the expression for the potential at a point P on the x-axis a distance x_p from the origin due to an infinitesimal segment of charge contained in infinitesimal segment of length dx at distance x from the origin.

b. Summing up the contributions from all of the elements of charge, find the expression for the potential at P due to this line of charge.

Brief Answers:

1. See this module's *text*.

2. See Problem 5 in this module's *Problem Supplement*.