PHYSNET

MISN-0-135

CAPACITANCE AND CAPACITORS

CAPACITANCE AND CAPACITORS

by

William F. Faissler

Title: **Capacitance and Capacitors**

Author: William Faissler, Dept. of Physics, Northeastern University, Boston, MA 02115.

Input Skills:

1. Apply Gauss's law to determine the electric field produced by a given planar, spherical, or cylindrical static charge distribution (MISN-0-132, MISN-0-133).
2. Derive the expression for the potential difference between the plates of a parallel plate capacitor (MISN-0-134).
3. Describe how one reduces a complex circuit having one type of element to a single element using the concept of *equivalent elements* and illustrating with an example (MISN-0-119).

Output Skills (Knowledge):

K1. Vocabulary: capacitance, capacitor, dielectric, dielectric constant, farad.

K2. Define capacitance in terms of charge and potential.

K3. Use Gauss's law to derive the capacitance of cylindrical and parallel plate capacitors.

K4. Derive, from energy and charge conservation principles, the capacitance of capacitors connected in series.

K5. Derive, from energy and charge conservation principles, the capacitance of capacitors connected in parallel.

Output Skills (Rule Application):

R1. Calculate the capacitance of a given cylindrical or parallel plate capacitor.

Output Skills (Problem Solving):

S1. Calculate the capacitance of a given series and/or parallel combination of capacitors.

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

CAPACITANCE AND CAPACITORS

by
William F. Faissler

1. Introduction

1a. Why We Study Capacitors. This module deals with capacitance, capacitors and a number of related topics. Although it may appear to you as if capacitance and capacitors are simply a digression in the process of understanding the electrostatic field, capacitors are one of the more important building blocks used in constructing modern electronic equipment. Thus the material you learn from this module is actually useful in dealing with electronic circuitry.

1b. Capacitance Defined. Two conductors separated from each other by empty space or by an insulator form a capacitor. Thus the word "capacitor" refers to a device consisting of the two conductors and whatever is in the space between them. A good example is the ordinary "Cable Television" cable. In this module it will be shown that if one of a capacitor's conductors has a charge of $+Q$ on it and the other has a charge of $-Q$, then the potential difference, V, between the two conductors is linearly proportional to the charge:

$$V = Q/C. \tag{1}$$

The proportionality constant's inverse, C, is called the "capacitance" of the capacitor. The numerical value of C depends on the sizes of the two conductors, on their relative positions, and on what is between them. It is independent of both Q and V.

1c. A Summary of What Follows. In this module we will derive the following results, except that we will assume vacuum between the plates of capacitors so $K = 1$:

1. For a parallel plate capacitor, the type commonly found in electronic circuits:

$$C = \frac{KA}{4\pi k_e d} \tag{2}$$

2. For a cylindrical capacitor, exemplified by a coaxial cable:

$$C = \frac{KL}{2k_e \ell n\,(R_2/R_1)} \tag{3}$$

3. For any capacitor, the voltage-current relationship:

$$i = C\frac{dV}{dt} \tag{4}$$

4. For any capacitor, the energy required to charge it:

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2 \tag{5}$$

5. For a number of capacitors in parallel, the equivalent capacitance:

$$C_{EQ} = C_1 + C_2 + \ldots + C_n. \tag{6}$$

and for the capacitors in series:

$$\frac{1}{C_{EQ}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n} \tag{7}$$

Finally, in the last section of the module, the effect of putting a dielectric material between a capacitor's conductors (making $K \neq 1$) will be discussed.

2. Capacitors

2a. The Parallel Plate Capacitor. The prototype of all capacitors is the parallel plate capacitor illustrated in Figure 1: we will here show that its capacitance is $C = A/(4\pi k_e d)$. This capacitor consists of two parallel conducting plates, each with area A and separated by a distance d. For the present this gap is assumed to be empty (vacuum); later in the module we will discuss what happens in the other usual case where
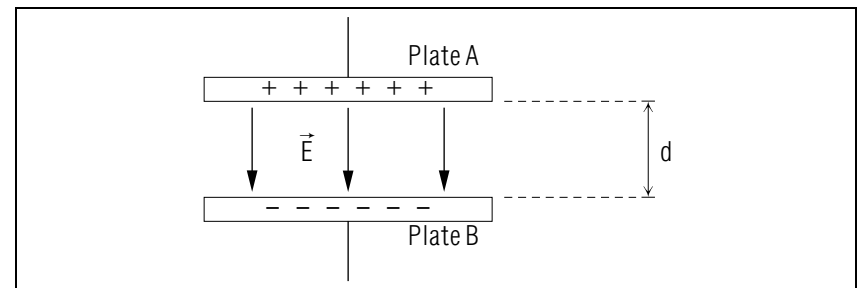

Figure 1. A parallel plate capacitor.

an insulating dielectric material is put between the two plates. One of the plates has a charge of $+Q$ while the other has a charge of $-Q$ and in this particular case we have assumed that the upper plate is positive while the lower plate is negative.

▷ By applying Gauss's law, you can easily see that the magnitude of the electric field in the gap between the two plates is:[1]

$$E = 4\pi k_e \frac{Q}{A} \tag{8}$$

and the direction of the electric field is from the upper plate to the lower plate as shown in the figure.

Next you can calculate the potential difference between the two plates, here labeled A and B:

$$V = -\int_A^B \vec{E} \cdot d\vec{\ell} = \int_A^B 4\pi k_e \frac{Q}{A} \, d\ell \,, = 4\pi k_e \frac{Qd}{A}$$

Thus the voltage across the capacitor is proportional to the charge on the capacitor, with the constant of proportionality being a combination of geometrical factors. This can be written as $V = Q/C$ [Eq. (1)] where the capacitance C is given by:

$$C = \frac{A}{4\pi k_e d} \qquad \text{(parallel plate)} \tag{9}$$

You recognize this as the result advertised earlier [Eq. (2)] for the case of vacuum ($K = 1$); later we will discuss the case of a capacitor made with an insulator other than vacuum.

2b. The Cylindrical Capacitor. The very common "cylindrical" capacitor is illustrated in Fig. 2. It consists of two concentric cylinders of length L, separated by a gap. For now it is empty space. One of the cylinders has a charge of $+Q$ on it and the other has a charge of $-Q$; in this particular case, the positive charge has been put on the inner cylinder.

▷ By applying Gauss's law you can easily see that the magnitude of the electric field in the gap between the two cylinders is:

$$E = 2k_e \frac{Q}{Lr} \qquad (R_1 \le r \le R_2) \tag{10}$$

---

[1] See "Gauss's Law Applied to Cylindrical and Planar Charge Distributions" (MISN-0-133).
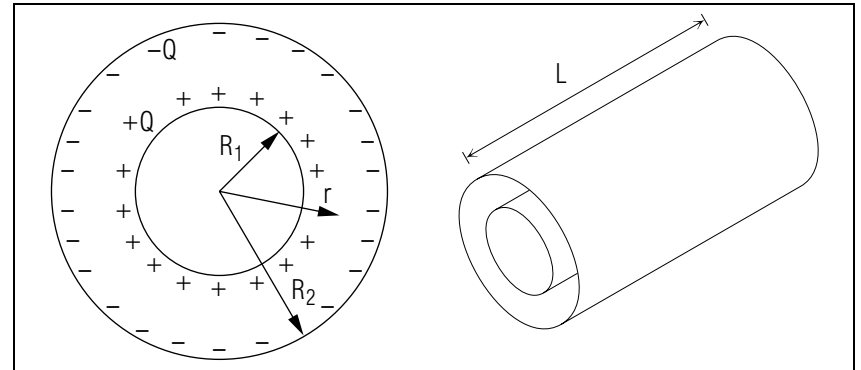
Figure 2. A cylindrical capacitor.

where r is the radius of the point at which you are calculating the field. Next you can calculate the potential difference between the inner and outer cylinders:

$$\begin{aligned} V &= -\int_A^B \vec{E} \cdot d\vec{r} = \int_{R_1}^{R_2} 2k_e \frac{Q}{L} \frac{dr}{r} \\ &= 2k_e \frac{Q}{L} (\ell n \, R_2 - \ell n \, R_1) \\ &= 2k_e \frac{Q \ell n \, (R_2/R_1)}{L} \end{aligned} \tag{11}$$

Once again, the voltage is proportional to the charge on either cylinder and the constant of proportionality is a collection of geometrical factors. Again writing $V = Q/C$ we find:

$$C = \frac{L}{2k_e \ell n \, (R_2/R_1)} \tag{12}$$

2c. A Generalization. In applying Gauss's law to the two configurations considered so far, it was necessary to assume that the point at which the electric field was being calculated was so far from the edges of the plates or cylinders that there was no effect due to the variation of the fields near the capacitor's edges. But what happens when the capacitor plates are so small or the cylinders so short that this cannot be true? It can be shown, although it is beyond the scope of this module to do so, that the form shown in Eq. (1) is true in all cases. For any arrangement of two isolated conductors with one having a charge of $+Q$ and the other

$-Q$, the potential difference between the two conductors is proportional to Q and the constant of proportionality $1/C$ depends only on geometrical factors. For the simple cases of two parallel plates or two concentric cylinders, Eq. (9) or (12), respectively, can be used to calculate the capacitance. For other, less idealized, geometries more complex formulae must be used. In any case, the capacitance C can be measured by electronic means and the validity of Eq. (1) is well established by experiment.

2d. The Units of Capacitance. Capacitance is measured in units of farad (abbreviated "F"); from Eq. (1) you can see that if one coulomb of charge is placed on a one farad capacitor, there will be a potential of one volt across the capacitor. For most electronic uses, the farad is a very large unit; commonly used capacitors are measured in units of microfarads, nanofarads and even picofarads.

▷ If you rearrange Eq. (1) you will find that capacitance has the dimensions of "charge per unit voltage" (voltage = electrostatic potential difference).

2e. Capacitors Are Neutral. As illustrated in Fig. 1, the normal capacitor as a whole is electrically neutral; one plate of the capacitor has a charge of $+Q$ on it while the other has a charge of $-Q$.

This sometimes creates a semantic problem, since capacitors are commonly said to "store charge." If you are concentrating on only one plate of the capacitor, then you may well be able to treat the capacitor as if it is "storing charge" but in general the capacitor as a whole is electrically neutral. When we say, "a capacitor has a charge Q on it," we mean that it has a charge of $+Q$ on one plate and $-Q$ on the other.

2f. The Voltage-Current Relation in a Capacitor. To find out what happens when the applied voltage changes with time, we simply rearrange Eq. (1) and differentiate it:

$$Q = C\,V$$

$$\frac{dQ}{dt} = V\frac{dC}{dt} + C\frac{dV}{dt}$$

$$\frac{dQ}{dt} = C\frac{dV}{dt}\,.$$

The last equation follows because C, which depends only on geometrical constants, is not a function of time so $dC/dt = 0$. Finally, dQ/dt (the rate of change of the charge on one of the plates) is the current flowing
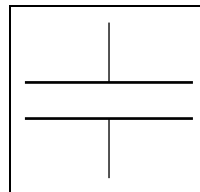
Figure 3. The symbol for a capacitor.

in the lead to that plate:

$$i = C\frac{dV}{dt}$$

This relationship is often used in determining the effects of capacitors in electronic circuits.

2g. The Symbol for a Capacitor. The symbol for a capacitor, to be used in drawing schematic diagrams, is illustrated in Fig. 3.

3. Useful Relationships

3a. The Energy Stored in a Charged Capacitor. It is often useful to know the amount of potential energy stored in a charged capacitor. Consider a capacitor C having charge of q. The potential difference between the two plates is given by Eq. (1): $V = q/C$. If a small amount of charge, dq, is moved from one plate to the other, then the amount of work done is:

$$dW = V\,dq = q\,dq/C\,. \tag{13}$$

Moving charge from one plate to the other is equivalent to any other method of charging the capacitor. Thus if you simply integrate Eq. (13), starting with zero charge and ending with a final charge Q, you will have calculated the work needed to charge the capacitor and hence you will have calculated the potential energy stored in the charged capacitor:

$$U = \int_O^Q q\,dq/C = \left.\frac{q^2}{2C}\right|_0^Q$$

$$= \frac{Q^2}{2C} = \frac{1}{2}VQ = \frac{1}{2}CV^2$$

where the last results follow by using Eq. (1).
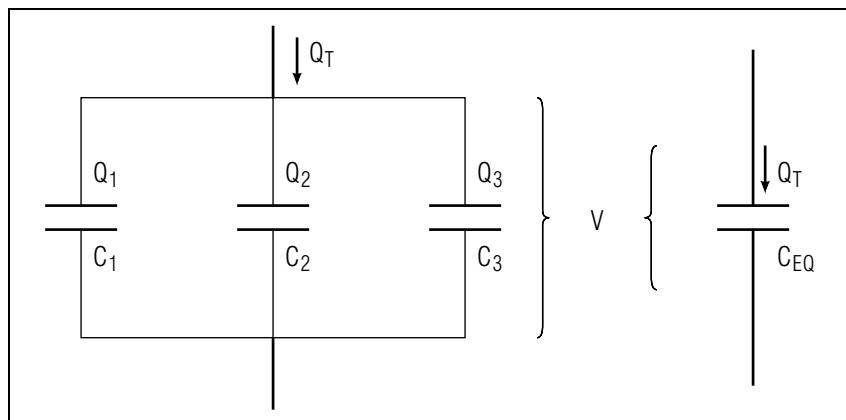
Figure 4. Three capacitors connected in parallel and the equivalent capacitor.

3b. Capacitors in Parallel. If two or more capacitors are in parallel, as in Fig. 4, they can be replaced by one equivalent capacitor in any electric circuit. In this parallel arrangement the same potential is necessarily applied to all capacitors:

$$V = \frac{Q_1}{C_1} = \frac{Q_2}{C_2} = \frac{Q_3}{C_3}$$

If you think about the process of charging the capacitor, you will be able to convince yourself that the total charge delivered down the common wire to the top plates of the capacitors is simply the sum of the charges on the individual top plates:

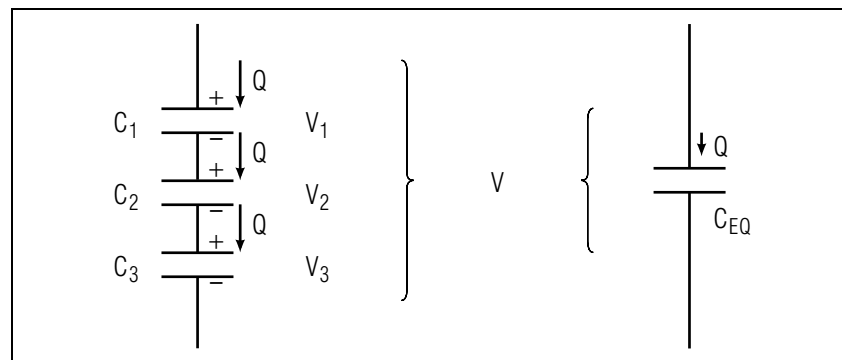$$Q_{\text{tot}} = Q_1 + Q_2 + Q_3$$

but of course

$$Q_{\text{tot}} = C_{\text{EQ}} V$$

so

$$C_{\text{EQ}} = C_1 + C_2 + C_3$$

If n capacitors are connected in parallel, this derivation can be generalized to show that the equivalent capacitance is given by:

$$C_{\text{EQ}} = C_1 + C_2 + \ldots + C_n.$$

Figure 5. The capacitor equivalent to three connected in series.

3c. Capacitors in Series. Just as in the parallel case, two or more capacitors in series can be replaced by one equivalent capacitor (see Fig. 5). If you start with all the capacitors uncharged and add a charge Q to the top plate of the top capacitor, then $+Q$ must leave the lower plate of the top capacitor, leaving it with a charge of $-Q$ if the capacitor is to remain neutral. The charge that leaves the lower plate of the top capacitor must end up on the top plate of the second capacitor, etc. Thus each of the capacitors ends up with the same charge Q on its top plate. The voltage across capacitor 1 is then:

$$V_1 = Q/C_1,$$

across capacitor 2 is

$$V_2 = Q/C_2,$$

and

$$V_3 = Q/C_3,$$

The total voltage across all three capacitors in the sum of V_1, V_2, and V_3:

$$V = V_1 + V_2 + V_3.$$

If the same charge were put on the equivalent capacitor, then the voltage across it would be

$$V = \frac{Q}{C_{\text{EQ}}}$$

Since this is the equivalent capacitor, one that can replace the three capacitors, then the two voltages must be the same. Combining these equations

gives:

$$\frac{Q}{C_{\text{EQ}}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

so

$$\frac{1}{C_{EQ}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

From the derivation above, it should be clear that if n capacitors are put in series, the equivalent capacitance would be

$$\frac{1}{C_{\text{EQ}}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}$$

4. [

Dielectric]Dielectrics

capacitor— with dielectric4. a.   Capacitors with Insulators If the two conducting plates of a capacitor are separated by an insulating material, a "dielectric," then it is found experimentally that the capacitance is K times as great as it is when the same plates are separated by vacuum. The constant K is called the "dielectric constant" of the insulating material. It does not depend on the potential difference across the capacitor or on the geometry of the capacitor. Table 1 contains a list of dielectric constants for common materials.

Since the dielectric constant of the insulating material is a multiplier on the capacitance of any capacitor, the capacitance of a parallel plate capacitor is, in general:

$$C = \frac{KA}{4\pi k_e d}$$

while the capacitance of a similarly insulated cylindrical capacitor is:

$$C = \frac{KL}{2k_e \ell n \, (R_2/R_1)}$$

Since the value of K for vacuum is 1 by definition of the units, Eqs. (9) and (12) are really just special cases of Eqs. (2) and (3).

4b. Electric Fields in Dielectrics.   The effects of dielectrics can be described in terms of the way they alter electric fields. If a dielectric is placed in a region of space where the electric field would otherwise have
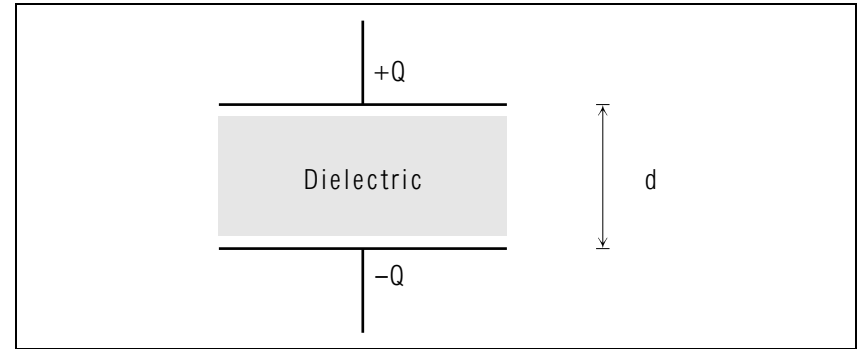
Figure 6. A parallel plate capacitor with dielectric between the plates.

a magnitude of E, and no other changes are made, then the magnitude of the electric field inside the dielectric is found to be reduced by a factor of K: it is E/K where K is again the dielectric constant. This fact makes it easy to understand the increase in the capacitance of the capacitor when the empty space is replaced by a dielectric. As an example, consider the parallel plate capacitor shown in Fig. 6.   If there were no dielectric, the electric field would be:

$$E = 4\pi k_e \frac{Q}{A} \, .$$

With the dielectric the field is:

$$E = 4\pi k_e \frac{Q}{KA}$$

The potential difference between the two plates is then:

$$V = 4\pi k_e \frac{Qd}{KA}$$

The capacitance is thus as in Eq. (2):

$$C = \frac{KA}{4\pi k_e d}$$

4c. An Atomic Model.   It is easy to construct an atomic model that makes plausible what is happening inside a dielectric subjected to an electric field. Each molecule of the dielectric material either already has an electric dipole moment or it develops one due to the applied electric
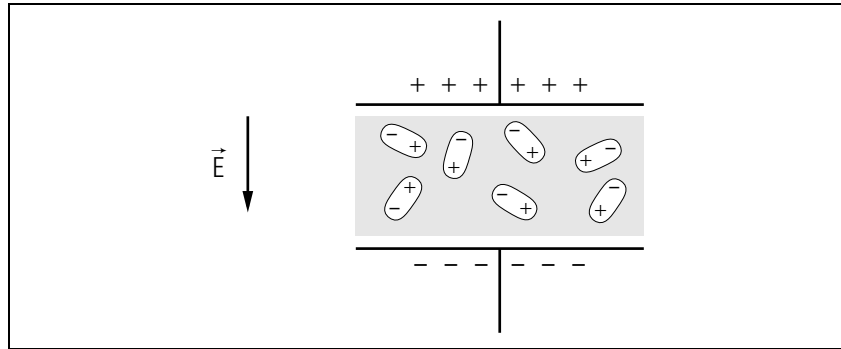
Figure 7. A dielectric in a capacitor.

field. Under the influence of the external electric field, these dipoles are aligned. The situation is shown in Fig. 7. These dipoles are aligned by the external electric field and as a result the electric field in the material is reduced. A more complete development of this model is the subject of another module.

4d. Dielectric Breakdown. For each type of dielectric there is a maximum electric field that can be applied to it before the dielectric breaks down and becomes a conductor. For some dielectrics this "break down" is a reversible process, as with insulating liquids, while for others it is irreversible, as with most circuit components. The maximum electric field that each dielectric can withstand is given in Table 1.

Table 1. Dielectric properties of several materials.		
Material	Dielectric constant	Breakdown field (Kv/cm)
Vacuum	1.0000	∞
Air	1.00054	8
Water	78.	*
Paper	3.5	140
Mica	5.4	1600
Porcelain	6.5	40
Fused Quartz	3.8	80
Pyrex Glass	4.5	130
Polyethylene	2.3	500
Polystyrene	2.6	250
Teflon	2.1	600
Pyranol oil	4.5	120
Titanium dioxide	100	60

*conducting

Glossary

- **capacitance**: the charge-storing capability of a capacitor, dependent on the geometrical configuration of the capacitor.

- **capacitor**: an electrical device that stores equal and opposite charge on two conducting surfaces separated by an insulator.

- **dielectric**: an insulating material.

- **dielectric constant**: a numerical factor that describes the amount to which a dielectric material increases the capacitance of a capacitor.

PROBLEM SUPPLEMENT

$$k_e = 8.99 \times 10^9 \, \mathrm{N\,m^2\,C^{-2}}$$

Note: Problems 9 and 10 also occur in this module's *Model Exam.*

1. A capacitor is connected to a 150 V voltage source and is charged until it holds 3.2×10^{-8} C. Without being discharged it is then connected to a 450 V source. How much additional charge is finally stored on the capacitor?

2. A 5.7 nF capacitor consists of two parallel square plates separated by 0.35 mm. How much charge is stored on the plates when a potential difference of 230 V is applied to the capacitor? With the capacitor still connected to the voltage source the plate separation is decreased to 0.25 mm. What is the new value for the charge stored in the capacitor?

3.
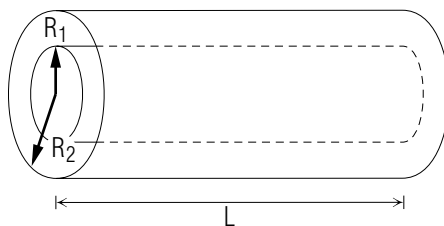
$R_1 = 4.5\,\mathrm{cm}$

$R_2 = 6.7\,\mathrm{cm}$

$L = 23.8\,\mathrm{cm}$

A particular cylindrical capacitor consists of two coaxial cylindrical metal shells with the dimensions given in the figure above. Compute the capacitance of the system.

4. In Sec. 2d of the text a statement was made that the unit of capacitance, the farad, is a very large unit. In order to begin to get a feeling for how large one farad is, suppose you had a parallel plate capacitor with square plates separated by a 1 mm air gap. If the capacitance is 1 F, how wide must the plates be? Also, calculate the width of the plates for $1\,\mu\mathrm{F}$, 1 nF, and 1 pF capacitors.

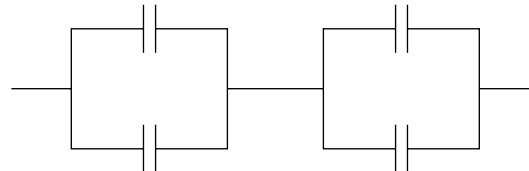5. Compute the equivalent capacitances for the following arrangements of four capacitors, each of 8.0 pF capacitance:
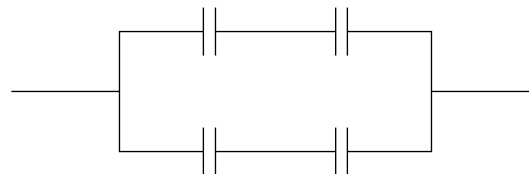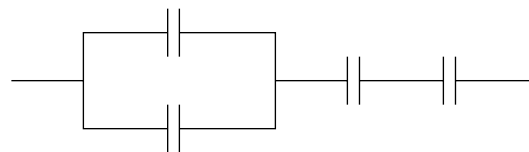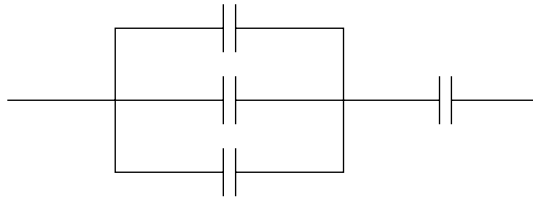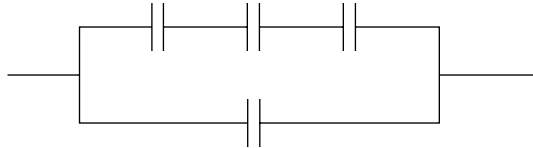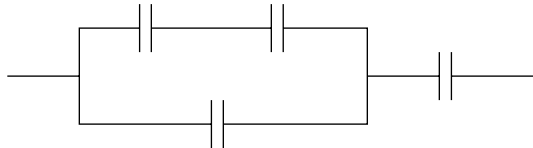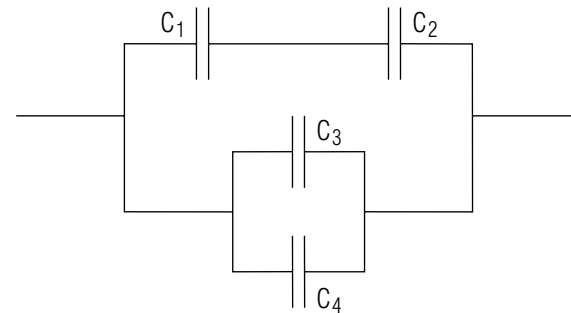
a.

b.
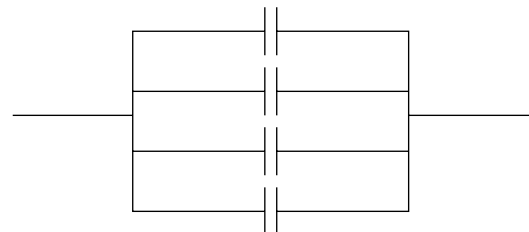
c.

d.

e.

f.

g.

h.

6. An electrical engineer needs a $25\,\mu\text{F}$ capacitor for a "breadboard" circuit he is working on. Although the shop's stockroom is temporarily out of $25\,\mu\text{F}$ capacitors, they do have $20\,\mu\text{F}$ capacitors. The engineer takes 20 of these capacitors and proceeds to construct a series and/or parallel combination of $20\,\mu\text{F}$ capacitors until the equivalent capacitance is $25\,\mu\text{F}$. Sketch a diagram that illustrates the combination of capacitors.

7. Consider a parallel plate capacitor with plates separated by a $1.25\,\text{mm}$ air gap, each with a surface area of $6.00\,\text{m}^2$. The capacitor is charged with a potential difference of $120\,\text{V}$ across the plates. How much energy is stored in the capacitor? The capacitor is disconnected from the voltage source so the charge is now trapped on the plates of the capacitor. With one plate fixed, the other plate is moved away from the fixed plate, increasing the separation of the plates to $1.50\,\text{mm}$. What is the new value for the amount of energy stored in the capacitor? Calculate the amount of work that must be done to increase the plate separation as stated. How is this quantity related to the energy stored in the capacitor, before and after the change in plate separation?

8. A cylindrical capacitor with air between the cylinders has a capacitance of $7.5\,\text{pF}$. The capacitor is connected to a constant voltage source and is completely charged. Without disconnecting the voltage source, pyranol oil is introduced between the cylindrical shells, completely displacing the air.

   a. Calculate the new capacitance of the capacitor.

   b. Does the energy stored in the capacitor increase or decrease?

   c. Does the energy stored increase or decrease if the capacitor is disconnected from the voltage source before adding the oil? Why or why not?

9. Compute the capacitance of a parallel plate capacitor with plates of area $25\,\text{cm}^2$ separated by $1\,\text{mm}$ of dielectric with a dielectric constant of 3.0.

10. Show the reduction to one capacitance that is equivalent to:
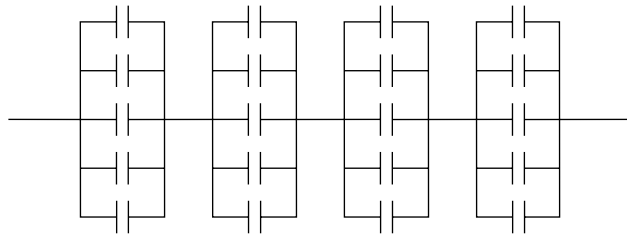
Brief Answers:

1. $Q = 6.4 \times 10^{-8}\,\text{C}$

2. $Q = 1.3 \times 10^{-6}\,\text{C}$; $Q' = 1.8 \times 10^{-6}\,\text{C}$ *Help: [S-1]*

3. $C = 3.3 \times 10^{-11}\,\text{F} = 33\,\text{pF}$

4. The widths of the plates would be: $10\,\text{km}$, $10\,\text{m}$, $34\,\text{cm}$, and $1\,\text{cm}$, respectively

5. $C_{\text{eq}} =$ (a) $2.0\,\text{pF}$; (b) $32.0\,\text{pF}$; (c) $8.0\,\text{pF}$; (d) $8.0\,\text{pF}$; (e) $3.2\,\text{pF}$; (f) $6.0\,\text{pF}$; (g) $10.7\,\text{pF}$; (h) $4.8\,\text{pF}$

6.
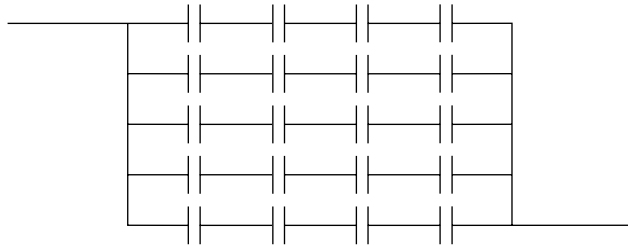
or

7. $U = 3.06 \times 10^{-4}\,\text{J}$; $U' = 3.67 \times 10^{-4}\,\text{J}$; $W = 6.1 \times 10^{-5}\,\text{J}$ *Help: [S-2]*; the increase in energy stored is equal to the work done on the capacitor.

8. a. $34\,\text{pF}$

   b. increases

   c. decreases; Q is now constant while C increases, and

$$U = \frac{1}{2}\left(\frac{Q^2}{C}\right).$$

9. $6.6 \times 10^{-11}\,\text{F} = 66\,\text{pF}$

10. $C_{\text{EQ}} = \dfrac{C_1 C_2}{C_1 + C_2} + C_3 + C_4$

SPECIAL ASSISTANCE SUPPLEMENT

S-1   *(from PS, problem 2b)*

$$Q' = \frac{(5.7 \times 10^{-9}\,\text{F})(0.35\,\text{mm})(230\,\text{V})}{0.25\,\text{mm}} = 1.8 \times 10^{-6}\,\text{C}$$

S-2   *(from PS, problem 7)*

As d changes, in this problem, Q cannot change so V changes as well as C. There are several different ways of obtaining the solution, some more succinct then others. Here is a final part of one solution (not a particularly succinct one):

$$W = \left(\frac{(6.00\,\text{m}^2)}{(2)(4\pi)(8.99 \times 10^9\,\text{N}\,\text{m}^2\,\text{C}^{-2})}\right) \times$$
$$\left(\frac{(144\,\text{V})^2}{1.50 \times 10^{-3}\,\text{m}} - \frac{(120\,\text{V})^2}{1.25 \times 10^{-3}\,\text{m}}\right)$$
$$= 6.1 \times 10^{-5}\,\text{J}.$$

MODEL EXAM

$$k_e = 8.99 \times 10^9 \, \mathrm{N\,m^2\,C^{-2}}$$

1. See Output Skills K1-K5 in this module's *ID Sheet*. One or more of these skills, or none, may be on the actual exam.

2. Compute the capacitance of a parallel plate capacitor with plates of area $25 \, \mathrm{cm}^2$ separated by $1 \, \mathrm{mm}$ of dielectric with a dielectric constant of 3.0.

3. Show the reduction to one capacitance that is equivalent to:
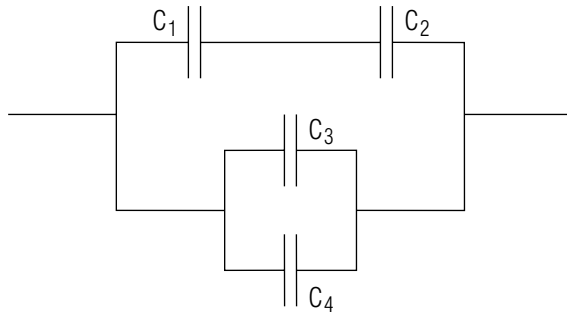
Brief Answers:

1. See this module's *text*.

2. See Problem 9 in this module's *Problem Supplement*.

3. See Problem 10 in this module's *Problem Supplement*.