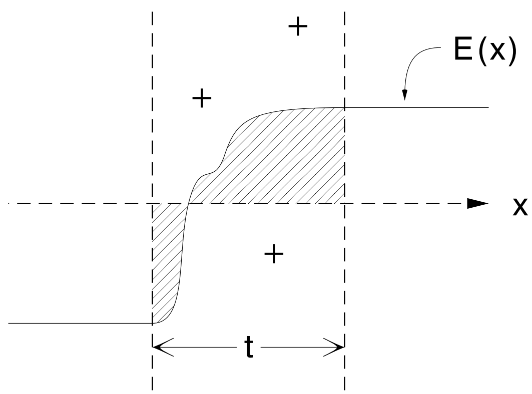
MISN-0-134

CHARGE LAYERS AND CAPACITORS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

CHARGE LAYERS AND CAPACITORS

by

Peter Signell

Title: **Charge Layers and Capacitors**

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Input Skills:

1. Vocabulary: surface and volume charge densities (MISN-0-147).
2. Compute the electrostatic potential difference between two points, given the electric field along any path connecting those two points (MISN-0-116).
3. Use Gauss's law to find the electric field produced by a given planar charge distribution (MISN-0-133).

Output Skills (Knowledge):

K1. Use Gauss's law to give a microscopic derivation of the fact of electric field discontinuity and potential continuity across a charge surface.

K2. Show graphically how the electric field and potential change while traversing a thin charge layer and a charge surface.

K3. Derive the electric field and potential functions for the region between the charge surfaces in parallel plate capacitors.

Output Skills (Problem Solving):

S1. Given some of the properties of a parallel plate air capacitor, calculate values for the other properties. Possible properties are: potential difference, electric field, plate area, distance between plates, plate charge and plate charge density.

Post-Options:

1. "Electrostatic Capacitance" (MISN-0-135).
2. "Electrostatic Field Energy" (MISN-0-137).

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New authors, reviewers and field testers are welcome.

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1. Introduction

In order to understand the common electronic circuit element called a capacitor, a metallic device used to temporarily store charge, we must examine the way that charge is stored and the electric fields and electric potential it produces. Fortunately, electrostatic charges introduced into a metal very quickly migrate to form a thin layer very close to the metal's surface. This charge layer is so thin that it can be safely approximated as a mathematical surface of zero thickness, resulting in a very simple description of the variation of the electric field and potential across it.

2. Field Across a Single Charge Layer

2a. The Thin Charge Layer. In order to connect the macroscopic (usual) and microscopic properties of a thin layer of charge, we must relate the charge distribution descriptors for those two cases. The ordinary and greatly enlarged cross-sectional views of a charge layer are shown in Fig. 1. Since no detail or structure can be seen in the edge-on macroscopic view, the appropriate description is the charge per unit surface area, the surface density σ, as seen from Regions 1 and 2 of the figure. Microscopically, however, we see a three-dimensional distribution of charge and so we describe it by the volume charge density ρ in a layer of thickness t. The two descriptions are related by $\sigma = \rho t$ as can easily be derived by evaluating the amount of charge seen within a definite area as viewed in each case from Region 1.[1]

2b. The Microscopic Field Across the Layer. The microscopic details of the electric field in the vicinity of the charge layer shown in Fig. 1 can be easily written down after mentally constructing appropriate Gaussian surfaces and then applying Gauss's law.[2] Applying Gauss's law twice, maintaining the appropriate x-direction symmetry for the Gaussian

[1] If the definite area is labeled S, the amounts of charge within it for the two cases are $\sigma S = \rho S t$.

[2] See "Gauss's Law Applied to Cylindrical and Planar Charge Distributions" (MISN-0-133).

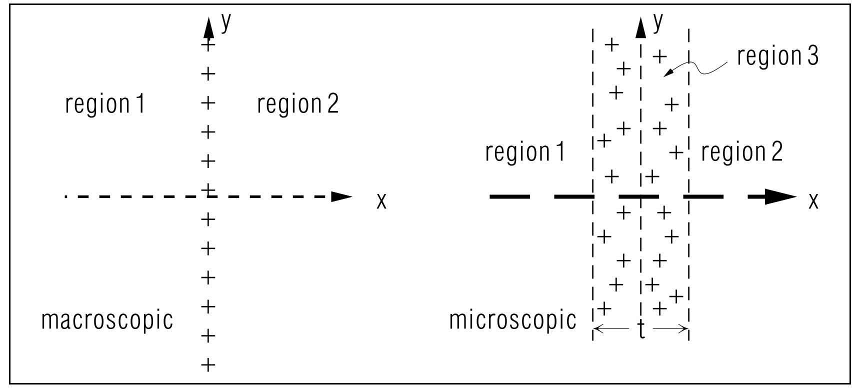

Figure 1. Two views of the same thin electric charge layer of thickness t.

surface each time, we obtain for each region of a layer centered on $x = 0$:

$$\begin{aligned} E_1(x) &= -2\pi k_e \sigma \,, \\ E_3(x) &= 4\pi k_e \rho x = 4\pi k_e \sigma \left(\frac{x}{t}\right) \,, \\ E_2(x) &= 2\pi k_e \sigma \,, \end{aligned} \qquad (1)$$

and these are plotted in Fig. 2. The force on a small test charge in each of the regions can be checked mentally, using the $E(x)$ line plotted in Fig. 2, to be sure that the Eqs. (1) are reasonable (i.e., positive charges repel a positive charge). Note also that $E(x)$ is continuous across the regional boundaries:

$$E_1\left(-\frac{t}{2}\right) = E_3\left(-\frac{t}{2}\right) \text{ and } E_3\left(\frac{t}{2}\right) = E_2\left(\frac{t}{2}\right).$$

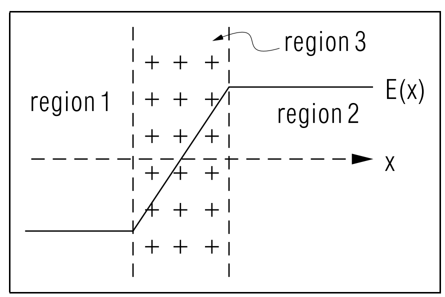

Figure 2. The electric field across a uniform volume charge density layer, assuming no other charges in the neighborhood. This is the same view as the "microscopic" one in Fig. 1.

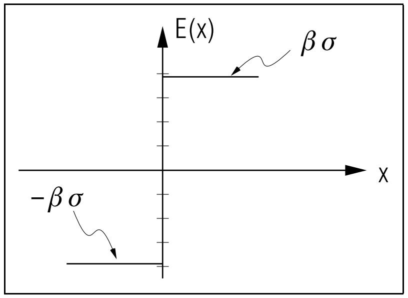

Figure 3. The apparent discontinuity in the electric field across a very thin charge layer. Substitute $\beta \equiv 2\pi k_e$.

2c. $E(x)$ Across a Charged Surface. The electric field actually varies continuously across a very thin layer of charge, but it appears to be discontinuous if we approximate the layer by a mathematical surface. The example of microscopic continuity shown in Fig. 2 can be taken out from under the microscope, so to speak, and then the thickness of the charged layer will be so small as to cause no discernible effect on the designated values of the electric field in the two outside regions (Regions 1 and 2 in Fig. 2): the resulting picture is shown in Fig. 3, where one can see that the discontinuity across a charged surface is:[3]

$$\Delta E_{\text{charge surface}} = 4\pi k_e \sigma\,. \tag{2}$$

3. Potential Across a Charge Layer

3a. Definition of V(x). Recall that the potential difference between two points is defined as the work per unit charge necessary to carry a small positive test charge from one point to the other.[4] Since we have to work against a positive electric field, the potential difference of two points along a one-dimensional field can be written (with a few steps left out):

$$\begin{aligned} \Delta V &= V(x_2) - V(x_1) = V_2 - V_1 \\ \Delta V &= -\int_{x_1}^{x_2} \vec{E}\cdot d\vec{x} = -\int_{x_1}^{x_2} E(x)\,dx. \end{aligned} \tag{3}$$

3b. ΔV Across a Charge Layer. No matter how the electric field varies inside a charge layer, its discontinuity between the layer's surfaces is always given by Eq. (2), $\Delta E = 4\pi k_e\sigma$. We will now show that the electrostatic potential drop across a charged surface is zero. To see this for the general case, we first construct an arbitrarily varying electric field, $E(x)$, corresponding to some arbitrarily varying volume charge density,[5] as in Fig. 4. The potential difference across the layer is seen by Eq.(3) to be the integral of $E(x)$ from one surface to the other. Since the integral of a function is just the net area under its graphical representation, the potential difference between the surfaces is just the net area under the $E(x)$ curve in Fig. 4. If we now say that the layer thickness t is very small, then the area will be very small and so will the potential difference. In fact, if we let the thickness go to zero in order to approximate the charge layer by a mathematical surface, the potential difference will obviously go to zero.

[3]To see that this equation holds for any charged surface, apply Gauss's law directly to the surface in Fig. 3.

[4]For further understanding of electrostatic potential and an analogy to the gravitational case, see "Electrostatic Potential Due to Discrete Charges" (MISN-0-116).

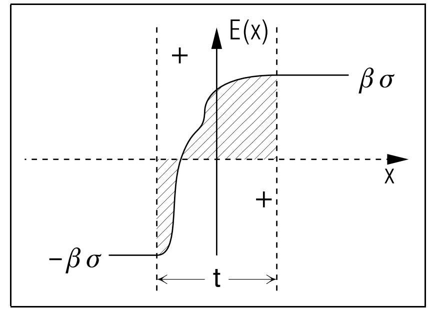

Figure 4. Electric field for an arbitrarily varying volume charge density in a layer of thickness t. Substitute $\beta \equiv 2\pi k_e$. The net shaded area is the potential difference between the edges.

3c. Two Other Ways of Seeing $\Delta V \to 0$ as $t \to 0$. It may seem strange that the work needed to carry a positive test charge across the surface goes to zero as the thickness t goes to zero, since the volume charge density ρ increases as $t \to 0$. Notice, however, that ρ is related to the derivative of the electric field, not to the field itself. The actual values of the field within the surfaces do not change[6] as we let $t \to 0$, so the force on our test charge at various internal points doesn't change. The force stays the same and the distance decreases so the work decreases.

[5]If the charge distribution is uniform in the y- and z-directions, then its volume charge density is given by the slope of $E(x)$: $dE(x)/dx = 4\pi k_e \rho(x)$.

[6]That is, the field at a fixed fractional value of the distance between the layer boundaries does not change. This is easily proved using Gauss's law.

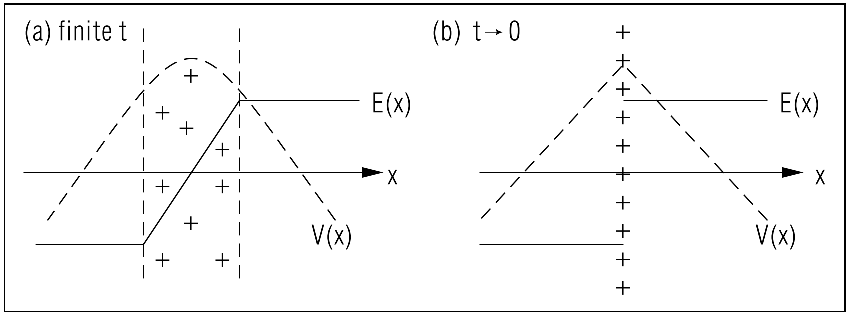

Figure 5. The effect on E and V as a uniform volume charge density layer is collapsed to zero thickness.

Another way of seeing the zero potential drop result is to use the calculus definition of the average of a function to write[7]

$$\Delta V = -\int_{x_1}^{x_2} E(x)\,dx = -[\bar{E}(x_1,x_2)]t.$$

where $\bar{E}(x_1,x_2)$ is the average electric field in the interval $x_1 \le x \le x_2$. As $t \to 0$ the average field stays constant (see Fig. 4), hence $\Delta V \to 0$.

3d. $V(x)$ and $E(x)$: Uniform Charge Density Example. Either by using the integral of Eq. (3), or by the differential slope relation

[7]Recall the definition of the average of a function $f(x)$ over the interval (x_1,x_2):

$$\bar{f}(x_1,x_2) \equiv \int_{x_1}^{x_2} \frac{f(x)\,dx}{(x_2 - x_1)}.$$

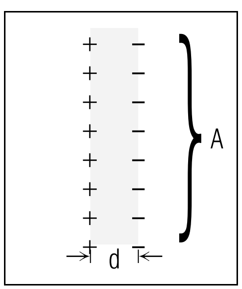

Figure 6. Schematic representation of a capacitor's cross section. Surface area is A, thickness is d.

and[8]

$$\frac{dV(x)}{dx} = -E(x),$$

we can construct the $V(x)$ corresponding to $E(x)$ for the case of uniform volume charge density within the layer. This is shown in Fig. 5a, and its appearance when collapsed into a mathematical surface is shown in Fig. 5b. Note that we can easily check the qualitative relationships between E and V at various x values by inspection.

4. $E(x)$ and $V(x)$ in Capacitors

A capacitor may be constructed from two parallel metallic plates separated by a layer of insulating material. The plates have equal but opposite charge, distributed on their surfaces, as shown in Fig. 6. We find $E(x)$ and $V(x)$ between the plates simply by combining the individual fields from each of the surfaces. The result is:

$$\begin{aligned} \vec{E} &= 4\pi k_e \sigma\,\hat{x} \quad && \text{for } 0 \le x \le d \\ &= 0 && \text{for other } x \end{aligned} \qquad (4)$$

and

$$\begin{aligned} V &= V_0 - 4\pi k_e \sigma\, x \quad && \text{for } 0 \le x \le d \\ &= 0 && \text{for other } x \end{aligned} \qquad (5)$$

where σ is the surface charge density on the positively charged plate. We can now integrate the electric field [as in Eq. (3)] and write down the potential difference between the plates:

$$V = 4\pi k_e \frac{Qd}{A}; \qquad C \equiv \frac{Q}{V} = \frac{A}{4\pi k_e d}\,, \qquad (6)$$

where Q is the total charge stored on the positively charged plate, A is the area of each plate, and C, defined for any system as Q/V, is called the system's "capacitance."[9] It is the system's ability to hold charge per unit potential difference, and it is the charge-storage capability used in designing electronic circuits.

[8]The following equation comes from factoring charge from both sides of the differential relation between potential energy and force. See "Potential Energy Curves, Motion, and Turning Points" (MISN-0-22).

[9]See "Electrostatic Capacitance" (MISN-0-135) where values are calculated for typical capacitor geometries as well as for combinations of capacitors.

Acknowledgments

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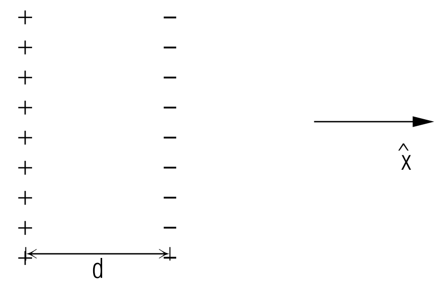
PROBLEM SUPPLEMENT

Note: Problem 6 also occurs in this module's *Model Exam.*

$$k_e = 8.99 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$$

1. For a parallel-plate capacitor with air between plates of surface area A and separation d, carrying total charges of $+Q$ and $-Q$:



   a. Use Gauss's law to find the electric field between the plates. *Help: [S-1]*

   b. Find the potential difference between the plates. *Help: [S-5]*

   c. Determine the capacitance.

2. A parallel-plate air capacitor having area $A = 3.0 \times 10^1\,\mathrm{cm}^2$ and spacing $d = 1.0\,\mathrm{mm}$ is charged to a potential difference of $5.00 \times 10^2\,\mathrm{V}$.

   a. Calculate its capacitance C.

   b. Find the magnitude of the charge on each plate.

   c. What is the value of the electric field between the plates?

3. A parallel-plate capacitor having circular plates of radius 10.0 cm and 1.5 mm spacing is charged to a potential difference of $2.00 \times 10^2\,\mathrm{V}$. It is then disconnected from the voltage source and the plates are pulled apart to twice the original separation. Calculate the value of the following quantities before and after the plates are pulled apart.

   a. the charge on the plates

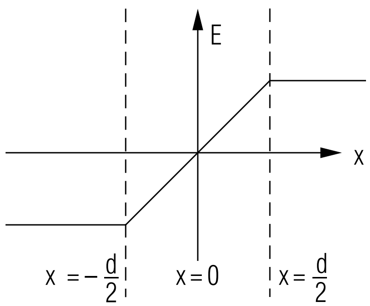
b. the electric field between plates

c. the potential difference between the plates

d. the capacitance

4. Use Gauss's law to determine the discontinuity ΔE in an electric field across a charged surface with surface charge density $\sigma = 5.0 \times 10^{-6}\,\mathrm{C/m^2}$? *Help: [S-4]*

5. The electric field varies inside a charge layer as shown below:



$$E(x) = \begin{cases} -E_0\,;\ x \leqslant -\frac{d}{2} \\ \frac{2E_0}{d}\,x\,;\ -\frac{d}{2} \geqslant x \geqslant \frac{d}{2} \\ E_0\,;\ x \geqslant \frac{d}{2} \end{cases}$$

Show that ΔV is zero in this case by:

a. calculating the net area under the $E(x)$ curve by geometrical means.

b. calculating the integral directly.

c. finding the integral using the average value of the electric field.

6. It has been experimentally shown that between the earth's surface and the region 50 km above the surface there is a potential difference of about 3×10^5 V. Calculate the surface charge density on the surface of the earth which is implied by these numbers. Note: Treat the "earth's surface to 50 km up" region as a parallel plate capacitor. [Data from "The Earth and Its Atmosphere as a Leaky...Capacitor," Martin Uman, **American Journal of Physics** - Vol. **42**, p. 1033 (1974).]

Brief Answers:

1. a. $\vec{E} = 4\pi k_e \dfrac{Q}{A}\,\hat{x}$

   b. $V = 4\pi k_e \dfrac{Qd}{A}$

   c. $C = \dfrac{A}{4\pi k_e d}$

2. a. $C = 2.7 \times 10^{-11}\,\mathrm{C/V}$ *Help: [S-2]*

   b. $Q = 1.3 \times 10^{-8}\,\mathrm{C}$

   c. $E = 5.0 \times 10^{5}\,\mathrm{V/m}$

3. a. $Q = 3.7 \times 10^{-8}\,\mathrm{C}$ before and after *Help: [S-3]*

   b. $E = 1.3 \times 10^{5}\,\mathrm{N/C}$,before and after

   c. $V = 2.0 \times 10^{2}\,\mathrm{V}$ before; $V = 4.0 \times 10^{2}\,\mathrm{V}$ after

   d. $C = 1.9 \times 10^{-10}\,\mathrm{C/V}$ before, $C = 9.3 \times 10^{-11}\,\mathrm{C/V}$ after

4. $\Delta E = 5.6 \times 10^{5}\,\mathrm{N/C}$

6. $5 \times 10^{-11}\,\mathrm{C/m^2}$

SPECIAL ASSISTANCE SUPPLEMENT

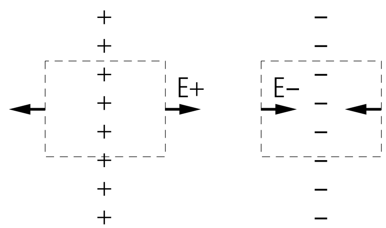
S-1 *(from PS, problem 1a)*

$(E_+)A + (E_+)A = 4\pi k_e(+Q)$

$(\vec{E}_+) = 2\pi k_e \dfrac{Q}{A}\,\hat{x}$

$(\vec{E}_-) = 2\pi k_e \dfrac{-Q}{A}\,(-\hat{x})$

$\vec{E} = (\vec{E}_+) + (\vec{E}_-) = 4\pi k_e \dfrac{Q}{A}\hat{x}$



S-2 *(from PS, problem 2a)*

$$C = \frac{(30)\left(10^{-4}\,\mathrm{m}^2\right)}{4\pi(8.99\times10^{9}\,\mathrm{N\,m^2\,C^{-2}})(0.001\,\mathrm{m})}$$
$$= 2.7\times10^{-11}\,\mathrm{C/V}$$

S-3 *(from PS, problem 3a)*

Reread the problem and see whether or not the battery is still connected to the plates while the "pulling apart" is going on.

S-4 *(from PS, problem 4)*

To use the data given, you must know how to get total charge from the area of a surface and the "surface charge density" on that surface [see Input Skill (1) in this module's *ID Sheet*].

S-5 *(from PS, problem 1b)*

The integrand, the electric field, is constant, so the integral is trivial.

MODEL EXAM

$$k_e = 8.99\times10^{9}\,\mathrm{N\,m^2\,C^{-2}}$$

1. See Output Skills K1-K3 in this module's *ID Sheet*.

2. It has been experimentally shown that between the earth's surface and the region 50 km above the surface there is a potential difference of about 3×10^{5} V. Calculate the surface charge density on the surface of the earth which is implied by these numbers. Note: Treat the "earth's surface to 50 km up" region as a parallel plate capacitor. [Data from "The Earth and Its Atmosphere as a Leaky...Capacitor," Martin Uman, **American Journal of Physics** - Vol. **42**, p. 1033 (1974).]

Brief Answers:

1. See this module's *text*.

2. See Problem 6, this module's *Problem Supplement*.