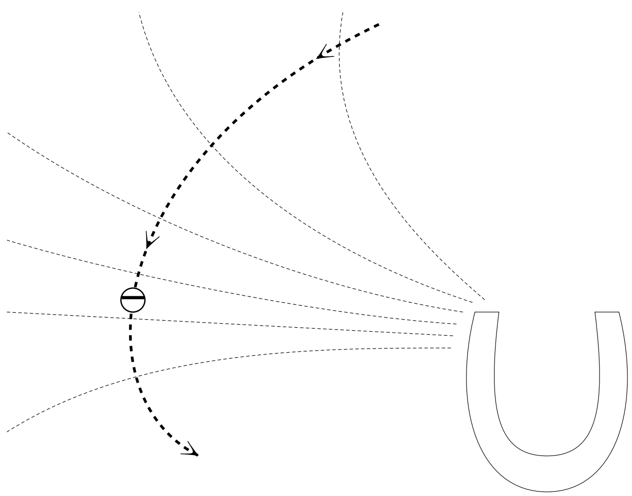
MISN-0-128

THE NUMEROV ALGORITHM FOR MAGNETIC FIELD TRAJECTORIES



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

THE NUMEROV ALGORITHM FOR
MAGNETIC FIELD TRAJECTORIES

by
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Title: **The Numerov Algorithm for Magnetic Field Trajectories**

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Input Skills:

1. Use the Taylor series to expand a function about a point (MISN-0-4).

Output Skills (Knowledge):

K1. Derive the recurrence relation for the Numerov Algorithm, to second order and in two dimensions, in a form suitable for use in obtaining the trajectory of a charged particle in an arbitrary magnetic field. Show all steps in the derivation.

K2. Derive equations for insertion of initial position and velocity in the Numerov Algorithm and communicate a method of obtaining a particular desired accuracy.

Post-Options:

1. "Trajectory of a Charged Particle in a Magnetic Field: Cyclotron Orbits (a computer project)" (MISN-0-127).

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1. Introduction and Description

Many problems in science and engineering can not be solved in terms of known functions, even when the under lying equation is known. Such a problem is the trajectory of a charged particle in a non-uniform magnetic field. For such cases one must resort to general numerical techniques: one of the most common is examined in this module.

2. Study Material

The force on a charged particle in a magnetic field is the Lorentz force:

$$\vec{F} = q\vec{v} \times \vec{B} \tag{1}$$

where q is the charge of the particle, $\vec{v}$ is its velocity, and $\vec{B}$ is the value of the magnetic field at the present location of the particle. The present force on the particle is $\vec{F}$. The force influences the particle's trajectory through Newton's Second Law:

$$\vec{F} = m\vec{a} \tag{2}$$

where m is the particle's mass.

In our case $\vec{B}$ will always be at right angles to $\vec{v}$, as is obvious from Eq. (1), hence $\vec{v}$ can change $\vec{B}$'s direction but not its magnitude. We will restrict ourselves to motion in the x-y plane by putting $\vec{v}$ there initially and putting $\vec{B}$ in the z-direction:

$$\vec{B} = B(x,y)\hat{z}$$

$$\vec{v} = x'\hat{x} + y'\hat{y} \tag{3}$$

where a prime denotes derivative with respect to time: $x' \equiv dx/dt$.

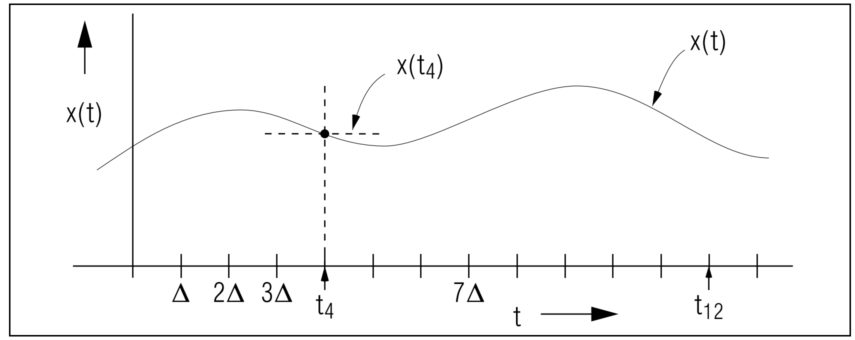

Figure 1. A function $x(t)$, specified at equally spaced values of t.

Equating the forces in Eqs. (1) and (2) and taking components (you do it) we get:

$$\begin{aligned} mx'' &= qy'B \\ my'' &= -qx'B \\ mz'' &= 0 \end{aligned} \tag{4}$$

Forget the third (z) equation since its solution does not couple to those of the x- and y-equations. Note that the x- and y-equations are "coupled", in that x'' involves y' and y'' involves x'. If we define $a(x,y) \equiv (q/m)B(x,y)$ then Eqs. (4) can be written:

$$x'' = ay'$$

$$y'' = -ax'$$

or, equivalently,

$$v'_x(t) = a(t)v_y(t); v'_y(t) = -a(t)v_x(t)\,. \tag{5}$$

In the Numerov method we deal with the solution functions $x(t)$, $y(t)$, $v_x(t)$, and $v_y(t)$, as a series of numbers at "net-point" times that are integrally spaced:

$$t_n = n\Delta\,.$$

This is illustrated in Fig. 1 for $x(t)$.

We then write:

$$
\begin{aligned}
x_n &\equiv x(t_n) \equiv x(n\Delta)\\
y_n &\equiv y(t_n) \equiv y(n\Delta)\\
v_{x,n} &\equiv v_x(t_n) \equiv v_x(n\Delta)\\
v_{y,n} &\equiv v_y(t_n) \equiv v_y(n\Delta)
\end{aligned}
$$

and our Eqs. (3) and (5) become:

$$
\begin{aligned}
x'_n &= v_{x,n}\\
y'_n &= v_{y,n}\\
v'_{x,n} &= a_n v_{y,n}\\
v'_{y,n} &= -a_n v_{x,n}\,.
\end{aligned}
\tag{6}
$$

These are four coupled equations.

We now connect the consecutive values of the x's and v's by making Taylor's Series expansions of each of them. For example:

$$x(t+\Delta) = x(t) + \Delta x'(t) + \frac{\Delta^2}{2!}x''(t) + \ldots$$

We will choose a sufficiently small so that terms beyond the second will be negligible compared to the first two terms. Then in our net-point notation and using Eq. (6):

$$
\begin{aligned}
x_{n+1} &= x_n + \Delta v_{x,n}\\
v_{x,n+1} &= v_{x,n} + \Delta a_n v_{y,n}
\end{aligned}
\tag{7}
$$

▷ You derive the equations for y_{n+1} and $v_{y,n+1}$.

Given the $t = 0$ position and velocity components,

$$x_0;\ y_0;\ v_{x,0};\ v_{y,0}$$

we can use the four "recurrence" relations (7) to generate the four position and velocity components at time $t = \Delta$:

$$x_1;\ y_1;\ v_{x,1};\ v_{y,1}\,.$$

Putting the latter back into the right hand side of the recurrence relations, we get the values at time $t = 2\Delta$. Continuing this process, we can find

the trajectory as far into the future as we wish. We must only be careful to put the correct value of $\vec{B}$ into a at each space-point (x_n,y_n).

Finally, how does one know what size time interval Δ to use? One could attempt to assess the importance of successive terms in the Taylor's Series, but a more reliable method is to decrease Δ until the predicted trajectory stabilizes; that is, until it does not change significantly when Δ is made even smaller. However, one must be aware that if Δ is continually made even smaller, a point will be reached where the errors will start increasing due to the computer's finite-word-size limit.

The algorithm, then, consists of:

1. recurrence relations
2. method of assuring desired accuracy
3. insertion of initial conditions.

Acknowledgments

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