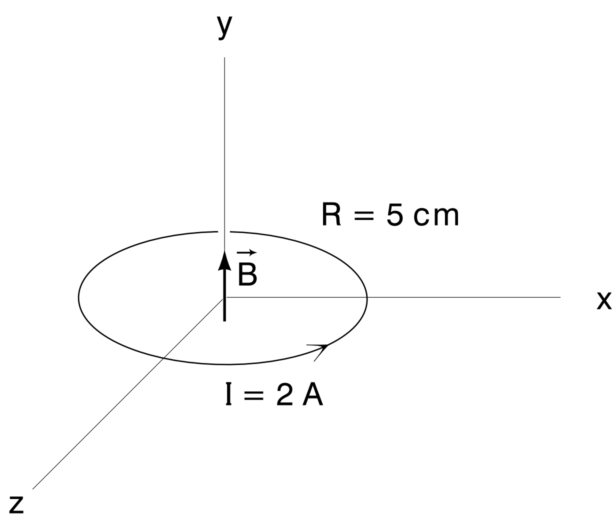
PHYSNET

MISN-0-125

THE AMPERE - LAPLACE - BIOT - SAVART LAW



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

THE AMPERE - LAPLACE - BIOT - SAVART LAW
by
Orilla McHarris and Peter Signell

1. Introduction 1

2. The Ampere-Laplace-Biot-Savart Law 1

3. Solutions
a. Overview 2
b. Obtaining the Point-Charge Equation 2
c. A Circular Loop of Current 3
d. A Long Straight Current-Carrying Wire 3

4. Two Parallel Wires 5

Acknowledgments 6

Glossary 6

Title: **The Ampere - Laplace - Biot - Savart Law**

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Version: 2/1/2000 Evaluation: Stage 1

Length: 1 hr; 20 pages

Input Skills:

1. Calculate the force on a straight segment of current-carrying wire in a magnetic field, given I, $\vec{B}$, and the length of the segment of wire (MISN-0-123).
2. State the relationship between the electric and magnetic fields set up by a moving point charge (MISN-0-124).
3. Derive the relationship $Q\vec{v} = I\vec{\ell}$ between total charge Q in length $\vec{\ell}$, moving with velocity $\vec{v}$, and constituting current I (MISN-0-123).

Output Skills (Knowledge):

K1. Vocabulary: Ampere-Laplace-Biot-Savart law.

K2. Start from the Ampere-Laplace-Biot-Savart law and obtain the equation for the magnetic field due to a point particle.

K3. Start from the Ampere-Laplace-Biot-Savart law and derive the equation for the magnetic field on the axis of a loop of current.

K4. Outline how one determines the magnetic field near a long straight wire using the Ampere-Laplace-Biot-Savart law.

Output Skills (Rule Application):

R1. Given the dimensions and current in a circular loop of current, determine the magnetic field at its center.

R2. Given the currents in two parallel wires, and the distance between the wires, calculate the force per unit length between the wires.

Post-Options:

1. "Magnetic Dipoles" (MISN-0-130).
2. "Ampere's Law" (MISN-0-138).

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New authors, reviewers and field testers are welcome.

THE AMPERE - LAPLACE - BIOT - SAVART LAW

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1. Introduction

In calculating the magnetic fields due to electric currents, it is sometimes easier to use Ampere's law and sometimes easier to use the Ampere-Laplace-Biot-Savart[1] law (hereafter called the ALBS law for short). For example, the magnetic field of a very long straight current is more easily obtained using Ampere's law, whereas the magnetic field of a loop of current is more easily found using the ALBS law. For cases in which the current does not follow a geometrically simple path, the ALBS law is the only feasible approach. The two laws both originate in Maxwell's Equations, but while Ampere's law is itself part of one Maxwell equation, the derivation of the ALBS law incorporates both Ampere's law and a second Maxwell equation. The magnetic field of a moving point charge is a special case of the ALBS law.[2]

In this module we will use the ALBS law to solve the case of a point charge, a loop of current, and a long straight wire, and then we will discuss the sign of the forces between adjacent current-carrying wires.

2. The Ampere-Laplace-Biot-Savart Law

The ALBS law for the magnetic field $\vec{B}$ due to a current I is:

$$\vec{B} = k_m\, I \int \frac{\hat{\mathcal{T}} \times \hat{r}}{r^2}\, d\ell\,. \qquad (1)$$

where k_m is the magnetic force constant, $d\ell$ is an element of length along the path taken by the current, $\hat{\mathcal{T}}$ is a unit vector tangent to the path at the element $d\ell$, and $\vec{r}$ is the position vector of the point at which $\vec{B}$ is being determined, as seen from the current element $d\ell$. Some of these relationships are illustrated in Fig. 1.

[1]Pronunciations: lä-pläss′, bee-oh′, sav-ar′.

[2]See "The Magnetic Field of a Moving Charge: Magnetic Interactions" (MISN-0-124).

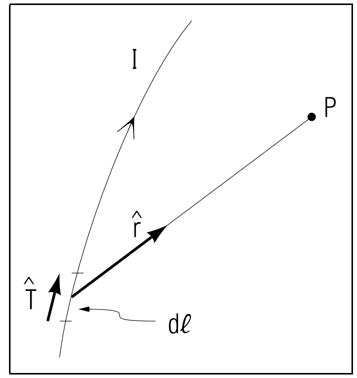


Figure 1. An infinitesimal segment of a current I of length $d\ell$ and with tangent $\hat{\mathcal{T}}$, and a point P at which we wish to know $\vec{B}$.

3. Solutions

3a. Overview. Equation (1) can be solved for the magnetic field at any space-point due to a current of any shape. In principle one need only integrate along the path followed by the charges in the current, putting in the appropriate values for the quantities in the integrand at each point on the path. For some currents having simple geometrical shapes the integral can be performed formally, and we shall examine several such cases in this module. For other cases, however, one must approximate the path of the current by tiny segments and sum the integrand over these segments using a computer.

3b. Obtaining the Point-Charge Equation. The idea here is to start with Eq. (1) and obtain the magnetic field due to a point charge q moving with velocity v.[3] To do this we must make an element of current I, of length $d\ell$ and direction $\hat{\mathcal{T}}$, describe a point charge q moving with velocity v. This can be done more elegantly using the Dirac delta function.[4] Here we assume that the particle's charge q is concentrated in such a *small length* ℓ that $\vec{r}$ does not change appreciably as one goes from one end of ℓ to the other. Since there is no current outside ℓ, which is the width of the charged particle, the integrand is zero everywhere except over ℓ. Then we can trivially integrate the essentially-constant integrand

[3]See "The Magnetic Field of a Moving Charge: Magnetic Interactions" (MISN-0-124).

[4]If interested, see "The Dirac Delta Function," MISN-0-380. If you are not familiar with this function, it may require a significant amount of studying before you feel comfortable with it.

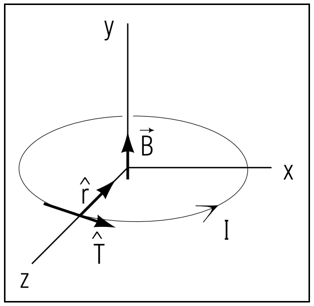


Figure 2. The magnetic field at the center of a circular loop of current.

over the length ℓ and get:

$$\vec{B} = k_m\, I\, \frac{\hat{\mathcal{T}} \times \hat{r}}{r^2}\, \ell\,.$$

Now we already know, for a point charge q of length ℓ and velocity v, that $I\hat{\mathcal{T}}$ is given by:[5]

$$Q\vec{v} = \ell\, I\, \hat{\mathcal{T}} \tag{2}$$

so we finally have the familiar equation for the magnetic field due to a moving point charge:

$$\vec{B} = k_m \frac{q\, \vec{v} \times \hat{r}}{r^2}\,. \tag{3}$$

3c. A Circular Loop of Current. We here calculate the magnetic field at the center of a circular loop of current lying in the x-z plane. First, notice that $\hat{\mathcal{T}}$ and $\hat{r}$ in Eq. (1) are always perpendicular and $\hat{\mathcal{T}} \times \hat{r} = \hat{y}$. The distance r is just the radius of the circle and the path of integration is the circumference of the circle (see Fig. 2):

$$\vec{B} = k_m I \int \frac{\hat{y}}{R^2}\, d\ell = k_m I \left(\frac{1}{R^2}\right)(2\pi R)\hat{y} = 2\pi k_m \frac{I}{R}\hat{y}\,. \tag{4}$$

Note that for this case, with the current traveling counterclockwise as seen from above, the field is in the positive y-direction at the origin. If the current had been clockwise, the direction of $\hat{\mathcal{T}}$ and hence of $\hat{B}$ would have been reversed.

3d. A Long Straight Current-Carrying Wire. Now we calculate the magnetic field near a long straight current-carrying wire located along the y-axis with current moving in the positive y-direction. First, notice

[5]See "Force on a Current in a Magnetic Field" (MISN-0-123).

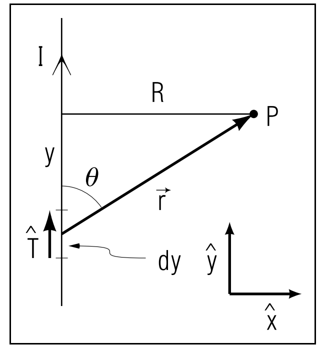


Figure 3. Calculating the magnetic field near a long straight current-carrying wire.

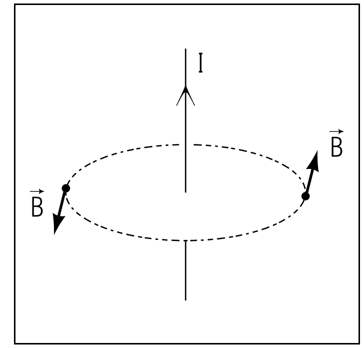


Figure 4. The magnetic field of Fig. 3 is tangential to an imaginary circle around the current.

that $\hat{\mathcal{T}} \times \hat{r}$ and r are not constant here as they were in the case of a circle. That means they must be written in terms of the variable of integration before they can be integrated. Since the integral extends the length of the y-axis, as it is drawn in Fig. 3, let us write all of the variables in terms of y for a point P that is in the plane of the page and to the right of the wire.

$$d\ell = dy\,, \tag{5}$$

$$\hat{\mathcal{T}} \times \hat{r} = -\hat{z}\sin\theta = -\hat{z}\, R(R^2 + y^2)^{-1/2}\,, \tag{6}$$

and

$$r = (R^2 + y^2)^{1/2}\,. \tag{7}$$

Then

$$\vec{B} = -\hat{z}\, k_m\, I\, R \int_{-\infty}^{+\infty} \left(R^2 + y^2\right)^{-3/2} dy\,. \tag{8}$$

We can look up this integral in a Table of Integrals and get

$$\begin{aligned} \vec{B} &= -\hat{z} k_m I R \frac{y}{R^2}\left(R^2 + y^2\right)^{-1/2}\Big|_{-\infty}^{+\infty}\,, \\ &= -\hat{z} k_m I R\left(\frac{2}{R^2}\right) = 2k_m \frac{I}{R}(-\hat{z})\,. \quad \textit{Help: [S-1]} \end{aligned} \tag{9}$$

The result, displayed in Fig. 4, is obtained more easily using Ampere's law.

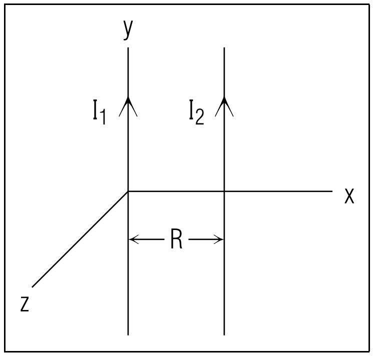


Figure 5. Two long parallel current-carrying wires.

4. Two Parallel Wires

From particle equations we were able to determine the direction of the force between two parallel current-carrying wires[6] but now, with the integrated forms, we can actually calculate the force. To make the equation easy to solve, we here assume that at least one of the wires is relatively long so that the magnetic field due to the ends is small compared to the magnetic field due to the long straight part. We know that the magnetic field due to a long straight wire along the y-axis, as shown in Fig. 5, is:

$$\vec{B} = -2k_m \frac{I}{R}\,\hat{z}\,, \tag{10}$$

and we know that the force on a current-carrying wire in a magnetic field[7] is $\vec{F}_B = I\vec{\ell}\times\vec{B}$. If we take I_1 as the source of the magnetic field and calculate the force on a wire carrying a current I_2 in the same direction as I_1 we have

$$\vec{F}_{2,1} = -I_2\vec{\ell}\times 2k_m\frac{I_1\hat{z}}{R} = -\hat{x}2k_m\frac{I_1 I_2 \ell}{R} \tag{11}$$

where the direction $-\hat{x}$ indicates the force is attractive. Often both wires are relatively long; in that case the quantity of interest is the force per unit of each wire on the other:

$$F/\ell = 2k_m\frac{I_1 I_2}{R}$$

where the force is attractive if I_2 is in the same direction as I_1, repulsive if I_2 is in the opposite direction from I_1.

[6]See "The Magnetic Field of a Moving Charge: Magnetic Interactions" (MISN-0-124).
[7]See "Force on a Current in a Magnetic Field" (MISN-0-123).

Acknowledgments

The author would like to thank Professor James Linneman for a helpful suggestion. The Problem Supplement was constructed by Kirby Morgan. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

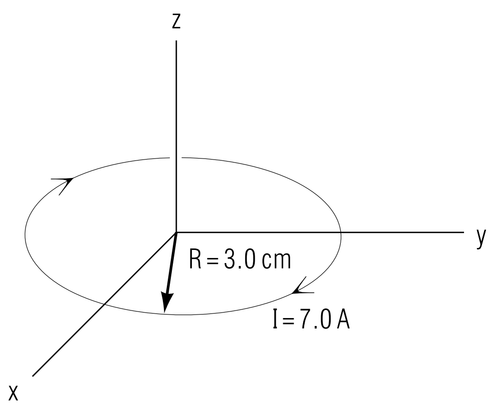
- **Ampere-Laplace-Biot-Savart law**: an integral formula that enables one to calculate the magnetic field at any space-point due to an electric current of any shape. Except for a few simple shapes for currents, however, a computer must be used to evaluate the law's integral.

PROBLEM SUPPLEMENT

$k_m = 10^{-7}\,\mathrm{N\,s^2\,C^{-2}}$

Note: Problems 6 and 7 also occur in this module's *Model Exam*.

1. Calculate the magnetic field at the center of a circular current loop that lies in the x-y plane as shown below.



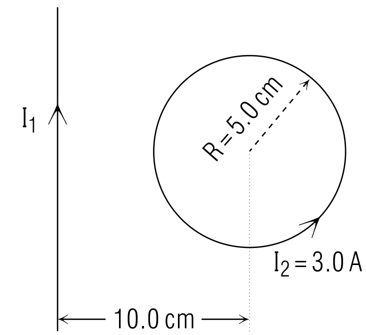
2. Two long parallel wires carry currents $I_1 = 0.020\,\mathrm{A}$ and $I_2 = 0.040\,\mathrm{A}$ in opposite directions and they are separated by a distance $R = 1.0 \times 10^1\,\mathrm{cm}$.

   a. Find the force per unit length on one wire due to the current in the other. *Help: [S-6]*

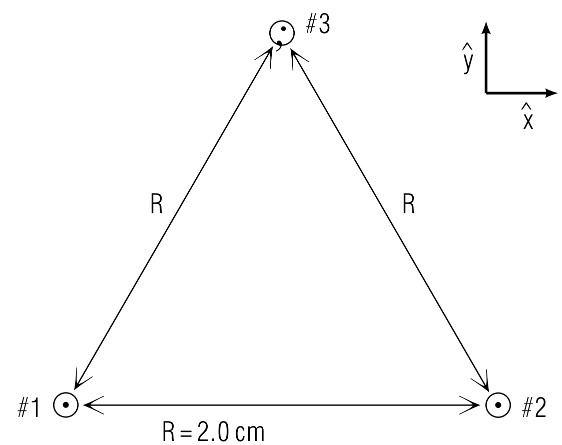
   b. Show whether the force is attractive or repulsive.

   c. Calculate $\vec{B}$ at a point half-way between the two wires.

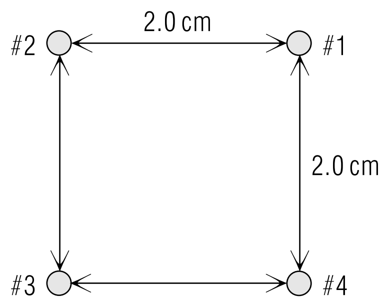
3. A long straight wire carries a current I_1 and lies in the same plane as a circular current loop as shown in the sketch. Find the magnitude and direction of I_1 such that the resultant magnetic field at the center of the loop is zero. State I_1 as positive if it is in the direction shown, as negative if in the opposite direction.



4. Three long parallel wires carry currents perpendicular to the plane of the page in the directions shown below. If each of the currents is 0.50 A, find the magnitude and direction of the force per unit length acting on wire #3.



5. Four long parallel wires each carrying a current of 0.10 A, are arranged at the corners of a square 2.0 cm on a side as shown in cross-section below.



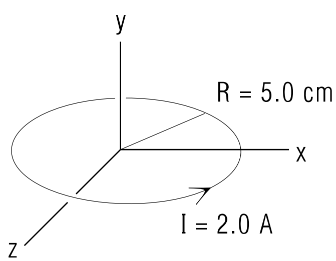
Find the magnitude of the magnetic field at the center of the square:

a. if all the currents are in the same direction (apply Fig. 4 twice). *Help: [S-4]*

b. if only three are in the same direction. *Help: [S-5]*

c. two are in one direction, the other two are in the opposite direction (two cases).

6.



Given a circular current loop in the x-z plane, integrate the Ampere-Laplace-Biot-Savart integrand and use the numbers in the diagram to find $\vec{B}$ at the center of the loop. Draw a diagram showing the directions of all vectors.

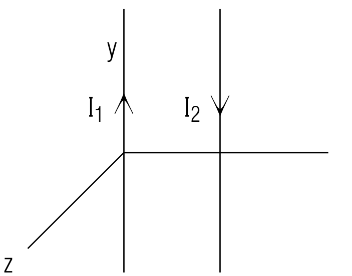
7. If a relatively long wire in your amplifier carried a current of 0.030 A, what would be the magnitude of the magnetic field at a transistor 4.0 mm away? What force would the wire exert on a parallel wire 3.0 mm away, 3.0 cm long and carrying a current of 0.040 A in the same direction?

Brief Answers:

1. $\vec{B} = -1.5 \times 10^{-4}\,\mathrm{T}\,\hat{z}$

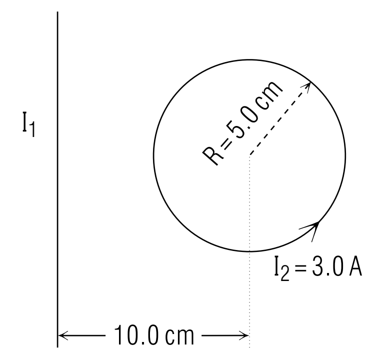
2. a. $F_\ell = 1.6 \times 10^{-9}\,\mathrm{N/m}$

   b. $F_{2,1} = \hat{x} 2 k_m \dfrac{I_1 I_2 \ell}{R}$ (force is repulsive)



   c. $\vec{B} = -2.4 \times 10^{-7}\,\mathrm{T}\,\hat{z}$

3. $I_1 = 18.8\,\mathrm{A}$, in direction shown. *Help: [S-2]*



4. $\vec{F}/\ell = 4.3 \times 10^{-6}\,\mathrm{N/m}\,\hat{y}$ *Help: [S-3]*

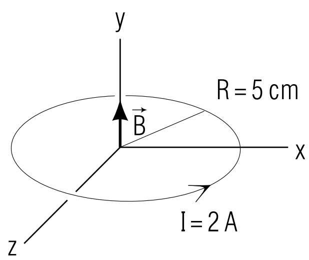
5. a. $B_{\mathrm{Total}} = 0$

   b. $B_{\mathrm{Total}} = 2.8 \times 10^{-6}\,\mathrm{T}$

   c. case 1: $B_{\mathrm{Total}} = 0$

   case 2: $B_{\mathrm{Total}} = 4.0 \times 10^{-6}\,\mathrm{T}$

6. $\vec{B} = (2.5 \times 10^{-5}\,\mathrm{T})\,\hat{y}$.



7. $B = 1.5 \times 10^{-6}\,\mathrm{T}$

$\vec{F} = 2.4 \times 10^{-9}\,\mathrm{N}$, toward the first wire.

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-3b)*

To take the limits $\ell = +\infty$ and $\ell = -\infty$ in this integral, realize that when

$$\ell \gg R,\ (R^2 + \ell^2)^{-1/2} \approx (\ell^2)^{-1/2} = \ell^{-1}$$

so

$$\lim_{\ell \to \infty} \frac{\ell}{R} \left(R^2 + \ell^2\right)^{-1/2} = \frac{\ell}{R|\ell|} = \frac{1}{R}.$$

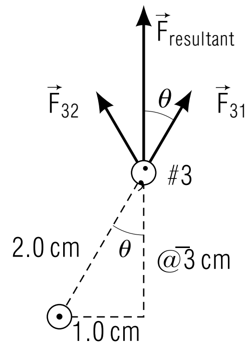
Similarly,

$$\lim_{\ell \to -\infty} \frac{\ell}{R} \left(R^2 + \ell^2\right)^{-1/2} = \frac{\ell}{R|\ell|} = -\frac{1}{R}.$$

S-2 *(from PS, problem 3)*

The fields due to the long straight wire and the current loop must be equal in magnitude and opposite in direction at the center of the loop. By applying the Ampere-Laplace-Biot-Savart law to a counterclockwise current loop, the magnetic field at the center of the loop is out of the page. Therefore you must decide, based on the directions involved in evaluating $\hat{T} \times \hat{r}$ for the Ampere-Laplace-Biot-Savart law as applied to a long, straight current-carrying wire, which direction for $\hat{T}$ (the direction of the current) will give a magnetic field into the page.
Numerical Help: [S-7]

S-3 *(from PS, problem 4)*



The forces on wire #3 due to wire #1 and wire #2 are both repulsive and directed as shown in the figure at right. The resultant force is the vector sum of the two equal magnitude forces

$$|\vec{F}_{31}| = |\vec{F}_{32}|;$$

$$\vec{F}_{\text{resultant}} = 2F_{31}\cos\theta\,\hat{y} = 2\,F_{32}\,\cos\theta\,\hat{y}$$

Numerical Help: [S-8]

S-4 *(from PS, problem 5a)*

Take the wires in diagonal pairs. Then you should be able to easily show that the $\vec{B}$ from one member of such a pair cancels the $\vec{B}$ from the other member of the same pair.

S-5 *(from PS, problem 5b)*

See [S-4] for one pair, but be able to argue your choice for the pair to use the technique of [S-4] on. Then use Fig. 3 to double the $\vec{B}$ from one wire of the remaining pair, and be able to argue why you can do this.

S-6 *(from PS, problem 2a)*

$$\frac{(2)(10^{-7}\,\text{N/A}^2)(0.02\,\text{A})(0.04\,\text{A})}{0.1\,\text{m}} = 1.6\times10^{-9}\,\text{N/m}$$

S-7 *(from [S-2])*

$(10\,\text{cm}/5\,\text{cm})(\pi)(3\,\text{A}) = 18.8\,\text{A}$

S-8 *(from [S-3])*

$$F/\ell = \frac{(2)(10^{-7}\,\text{N/A}^2)(\cos 30^\circ)(0.5\,\text{A}+0.5\,\text{A})(0.5\,\text{A})}{2\times10^{-2}\,\text{m}} = 4.3\times10^{-6}\,\text{N/m}$$

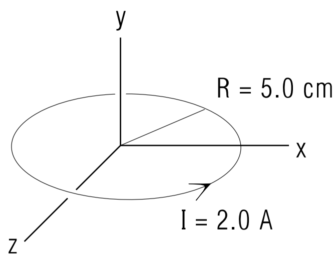
MODEL EXAM

$k_m = 10^{-7}\,\mathrm{N/A^2}$

$\vec{B} = k_m\, I \int \dfrac{\hat{\mathcal{I}} \times \hat{r}}{r^2}\, d\ell\,.$

1. See Output Skills K1-K4 in this module's it ID Sheet.

2.



Given a circular current loop in the x-z plane, integrate the Ampere-Laplace-Biot-Savart integrand and use the numbers in the diagram to find $\vec{B}$ at the center of the loop. Draw a diagram showing the direction of all vectors.

3. If a relatively long wire in your amplifier carried a current of 0.030 A, what would be the magnitude of the magnetic field at a transistor 4.0 mm away? What force would the wire exert on a parallel wire 3.0 mm away, 3.0 cm long and carrying a current of 0.040 A in the same direction?

Brief Answers:

1. See this module's *text*.

2. See Problem 6 in this module's *Problem Supplement*.

3. See Problem 7 in this module's *Problem Supplement*.