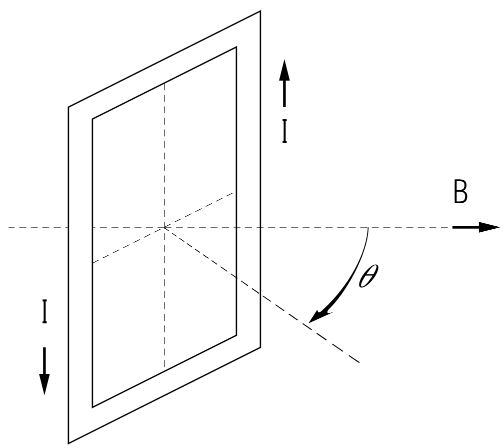
MISN-0-123

FORCE ON A CURRENT IN A MAGNETIC FIELD



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

FORCE ON A CURRENT IN A MAGNETIC FIELD

by

Orilla McHarris

Title: **Force on a Current in a Magnetic Field**

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Input Skills:

1. Vocabulary: current (MISN-0-117); magnetic field, magnetic force (MISN-0-122).
2. Calculate the torque on a rod with respect to a parallel axis, given a constant force perpendicular to the rod (MISN-0-5).
3. Visualize the rotational motion produced by a given torque (MISN-0-33).
4. Calculate the magnetic force on a moving charged particle (MISN-0-122).

Output Skills (Knowledge):

K1. Derive the expression $Q\vec{v} = I\vec{\ell}$ relating a set of charges and their common velocity to their equivalent value as a current.

K2. Starting from the Lorentz force, derive the expression for the force on a length current-carrying wire in a uniform magnetic field.

Output Skills (Problem Solving):

S1. Calculate the force on a straight section of a current-carrying wire in a given uniform magnetic field.

S2. Calculate the mechanical torque on a rectangular current loop in a given uniform magnetic field.

S3. Calculate the torque on an n-turn rectangular coil in a given uniform magnetic field.

Post-Options:

1. "The Magnetic Field of a Moving Charge: Magnetic Interactions" (MISN-0-124).
2. "The Magnetic Field of a Current: The Ampere-Laplace Equation" (MISN-0-125).

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1. A Straight Current

1a. Current Consists of Moving Charges. An electric current consists of moving charges. Thus, if a current is placed in a magnetic field, it will be subject to a magnetic force just as single moving charges are. The magnetic force on one charge q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is:[1]

$$\vec{F}_B = q\vec{v} \times \vec{B}. \tag{1}$$

The magnetic force on a current is just the sum of such forces on the current's component charges. Assuming these charges all travel with the same velocity:

$$\begin{aligned}\vec{F}_{\text{Total}} &= q_1\vec{v} \times \vec{B} + q_2\vec{v} \times \vec{B} + \ldots + q_n\vec{v} \times \vec{B} \\ &= (q_1 + q_2 + \ldots + q_n)\vec{v} \times \vec{B}, \\ &= Q\vec{v} \times \vec{B},\end{aligned}$$

where Q is the total charge moving with velocity $\vec{v}$.

1b. Relationship Among Q, $\vec{v}$, And I. In order to state the magnetic force in terms of the measured current, I, we must find the relationship of Q and $\vec{v}$ to I. The amount of charge dQ in a small length of conducting wire $d\ell$ is given by the overall charge per length, Q/ℓ times the length $d\ell$:

$$dQ = \frac{Q}{\ell}\, d\ell \tag{2}$$

If we hold a current measuring instrument at one point and watch a length $d\ell$ of charge go by, for all of $d\ell$ (and therefore, all of dQ) to pass the point takes a time

$$dt = d\ell/v \tag{3}$$

where v is the speed of the charges making up the current. Now the definition of current at a point is the amount of charge per unit time

[1]See "Force on a Charged Particle in a Magnetic Field" (MISN-0-122).

passing the point:

$$I = \frac{dQ}{dt} = \frac{(Q/\ell)d\ell}{d\ell/v} = \frac{Qv}{\ell}. \tag{4}$$

Remember, however, that the velocity of positive charges and the current are in the same direction, so actually:

$$\vec{I}\ell = Q\vec{v}. \tag{5}$$

For convenience, we will transfer the designation of the current's direction to the length of the straight section of current-carrying wire:

$$I\vec{\ell} = Q\,\vec{v}. \tag{6}$$

Remember that the direction of $\vec{\ell}$ is always to be taken in the direction of the current.

1c. Force on a Current. Now it is easy to state the magnetic force in terms of current. For a set of charges all moving with the same speed in the same direction,

$$\vec{F}_B = Q\,\vec{v} \times \vec{B}, \tag{7}$$

or:

$$\vec{F}_B = I\,\vec{\ell} \times \vec{B}. \tag{8}$$

Notice that, in a complete electrical circuit, $\vec{\ell}$ will have to have several different directions, so different sides of a current loop will in general have different forces on them. The total force on a current loop is then the sum of the force on its separate sides (and if the loop is made up of curved circle, for example, the total force would be the integral of the small elements of force $d\vec{F}$ acting on each small length $d\vec{\ell}$).

2. A Rectangular Current

2a. Plane of Loop Parallel to $\vec{B}$. Let us calculate what will happen to a rectangular loop of current in a uniform magnetic field $\vec{B}$. First let us take the simple case where the loop is in the x-y plane and $\vec{B}$ is in the x-direction. We must apply Eq. (8) to each of the four sides of the loop separately (see Fig. 1):

a. Side a: $\vec{F}_a = 0$ *Help: [S-3]*

b. Side b: $\vec{F}_b = -ILB\hat{z}$; $\vec{F}$ tends to push side b *into* the page *Help: [S-4]*

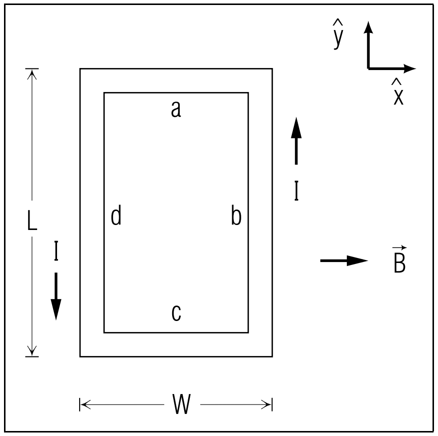


Figure 1. A rectangular current loop in a magnetic field parallel to the loop's width.

c. Side c: $\vec{F}_c = 0$

d. Side d: $\vec{F}_d = ILB\hat{z}$; $\vec{F}$ tends to push side d *out of* the page.

Thus the net result of the four forces is to produce a torque on the current loop about an axis through its center, parallel to the y-axis. We can calculate the torque about this axis:[2] *Help: [S-6]*

$$\begin{aligned} \vec{\tau} &= \sum_i \vec{\tau}_i = \sum_i \vec{r}_i \times \vec{F}_i = \vec{r}_a \times \vec{F}_a + \vec{r}_b \times \vec{F}_b + \ldots \\ &= [0 + \frac{W}{2} ILB + 0 + \frac{W}{2} ILB]\hat{y} \\ &= WLIB\hat{y} = (A\,I\,B)\,\hat{y} \end{aligned} \tag{9}$$

where $A = WL$ is the area enclosed by the loop.

Suppose the loop starts rotating in response to the torque, resulting in $\vec{B}$ no longer being in the plane of the loop. Then the equations derived above will no longer be valid because they assumed that $\vec{B}$ is in the loop plane.

2b. Loop at an Angle to $\vec{B}$. Since a torque on a loop will cause it to rotate, we now treat the case where such a rotation has produced an angle θ between $\vec{B}$ and the normal to the plane of the loop (when $\vec{B}$ is in the loop plane, $\theta = 90°$).[3] Figure 2a shows our rectangular current loop

[2] See "Force and Torque" (MISN-0-5).

[3] The "normal" to the plane of the loop is a unit vector perpendicular to the plane.

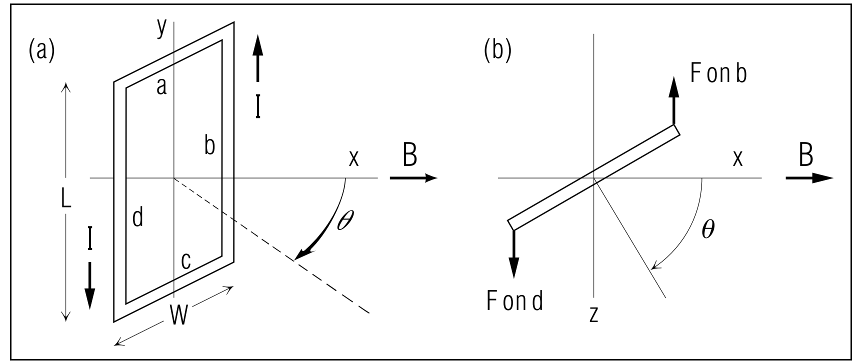


Figure 2. (a) Oblique view of a rectangular current loop whose normal is at an angle to $\vec{B}$; and (b) top view of the current loop.

with its normal rotated away from $\vec{B}$ by some angle θ. Figure 2b shows an overhead view of the same loop. Again we consider the force on each side separately:

a. Side a: $\vec{F}_a = IWB\sin(90° + \theta)\,\hat{y} = IWB\cos\theta\hat{y}$. The direction of $\vec{F}_a$ is such as to push side a and the entire current loop upward, in the positive y-direction.

b. Side b: $\vec{F}_b = I\,L\,B(-\hat{z})$. The direction of $\vec{F}_b$ is such to push side b backward, in the negative z-direction.

c. Side c: $\vec{F}_c = I\,W\,B\,\sin(90°-\theta)(-\hat{y}) = I\,W\,B\,\cos\theta(-\hat{y})$. The direction of $\vec{F}_c$ is such as to push c and the entire current loop downward, in the negative y-direction.

d. Side d: $\vec{F}_d = I\,L\,B\,\hat{z}$. The direction of $\vec{F}_d$ is such as to push d forward, in the positive z-direction.

Now notice that although the forces on sides a and c are no longer zero, they have equal and opposite effects and are radial; hence they have no effect other than a tendency to deform the loop if it is not rigid.[4] The forces on sides b and d still operate in such a way as to produce torques on the loop:

$$\vec{\tau}_b = \frac{W}{2} I\ell B\sin\theta\,\hat{y}\,,$$

[4] (1) Adding equal but opposite forces yields zero net force; and (2) any *radial* force produces zero torque.

$$\vec{\tau}_d = \frac{W}{2} I\ell B \sin\theta\, \hat{y}\,,$$

and thus the total torque is:

$$\vec{\tau} = W\,L\,I\,B\,\sin\theta\,\hat{y} = A\,I\,B\,\sin\theta\,\hat{y}\,.$$

It is apparent that the torque on the current loop is a maximum for $\theta = 90°$ (i.e. for $\vec{B}$ in the plane of the loop as in Fig. 1) and zero for $\theta = 0°$. Thus the tendency is for a current loop in a magnetic field to rotate until its normal is parallel to $\vec{B}$, and for it to decelerate as it shoots past that alignment. We could convert this oscillating loop into a rotating one and use it as a means of turning electrical energy into mechanical energy—that is, as a DC motor—if we could reverse the current in the loop just as its normal becomes parallel to $\vec{B}$.

2c. Generalization to a Current-Carrying Coil. It should be noted that it is a simple matter to generalize from the force or torque on a single turn loop of current to the force or torque on a many turn coil. In general, coils are wound with the area of each turn the same as all the others and of course the same current would flow through them all. Thus the force or torque on an n-turn coil is generally just n times the force or torque on a one-turn coil.

Acknowledgments

Kirby Morgan constructed the Problem Supplement. Mark Sullivan gave valuable feedback on an earlier version. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

A. Geometrical Definition of Vector Product

The vector product of two arbitrary vectors $\vec{A}$ and $\vec{B}$ is defined as the vector quantity whose magnitude is given by the product of the magnitudes of the two vectors times the sine of the angle between the vectors when they are placed "tail-to-tail," and whose direction is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$. The vector product (also referred to as the "cross product") is denoted by $\vec{A}\times\vec{B}$. The magnitude of the product may be written as:

$$|\vec{A}\times\vec{B}| = AB\sin\theta.$$

There are, however, two directions that are perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$. The correct direction may be chosen by applying the "right-hand rule": "Rotate vector $\vec{A}$ into vector $\vec{B}$ through the smaller angle between their directions when they are placed tail-to-tail. Follow this rotation with the curled fingers of your right hand, and the direction of your extended thumb identifies the direction of the vector product." This rule is sufficient to distinguish between the two possible choices for the direction of a vector product. Notice that the order of multiplication in vector products is very important. The product $B\times\vec{A}$ has the same magnitude as $\vec{A}\times\vec{B}$, but the directions of the two products are opposite. In general:

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}.$$

We say that vector products do not "commute," or that the vector product is a "noncommutative" operation.

If two vectors are parallel, the angle between their directions is zero, so by the definition of the magnitude of vector products their cross product is zero. Similarly, if two vectors are perpendicular, the angle between their directions is $90°$. Since $\sin 90° = 1$, the magnitude of the vector product of the two is just the product of their magnitudes, and the direction of the vector product is determined by the right-hand rule. By applying these observations to the vector product of the cartesian unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$, we may derive the following useful relations:

$$\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0$$

$$\hat{x}\times\hat{y} = -\hat{y}\times\hat{x} = \hat{z}$$

$$\hat{y}\times\hat{z} = -\hat{z}\times\hat{y} = \hat{x}$$

$$\hat{z}\times\hat{x} = -\hat{x}\times\hat{z} = \hat{y}.$$

These relations are used in the algebraic definition of vector products.

B. Algebraic Definition of Vector Product

If we express vectors $\vec{A}$ and $\vec{B}$ in their cartesian component form, the vector product of $\vec{A}$ and $\vec{B}$ may be written:

$$\vec{A}\times\vec{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}).$$

If this expression is expanded algebraically as we would the product $(x+3)\cdot(2x-5)$, except that the cross product is used instead of scalar multiplication, then this vector product may be expressed as a combination of the cartesian components of $\vec{A}$ and $\vec{B}$ and cross products of the cartesian unit vectors. Using the relations between the cartesian unit vectors developed in Appendix A and denoting the vector product as a third vector $\vec{C}$, we may write:

$$\vec{C} = \vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

or:

$$\begin{aligned} C_x &= A_y B_z - A_z B_y, \\ C_y &= A_z B_x - A_x B_z, \\ C_z &= A_x B_y - A_y B_x. \end{aligned}$$

The mnemonic for remembering the order of the subscripts on these components is to note that, starting from left to right, the first three subscripts in each of the three equations for the components of $\vec{C}$ are always cyclic permutations of xyz (xyz, yzx, zxy). Another way to remember the order of combination of the unit vectors and the components of $\vec{A}$ and $\vec{B}$ is to use this determinant:

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Expansion of this determinant leads to the same expression for $\vec{C}$ derived earlier.

$\triangleright$ Show that the vector product of $\vec{A} = 5\hat{x} - 2\hat{y}$ and $\vec{B} = \hat{x} + \hat{y} + 3\hat{z}$ has these components:

$$C_x = -6,\ C_y = -15,\ C_z = 7.$$

C. Direction of the Vector Product

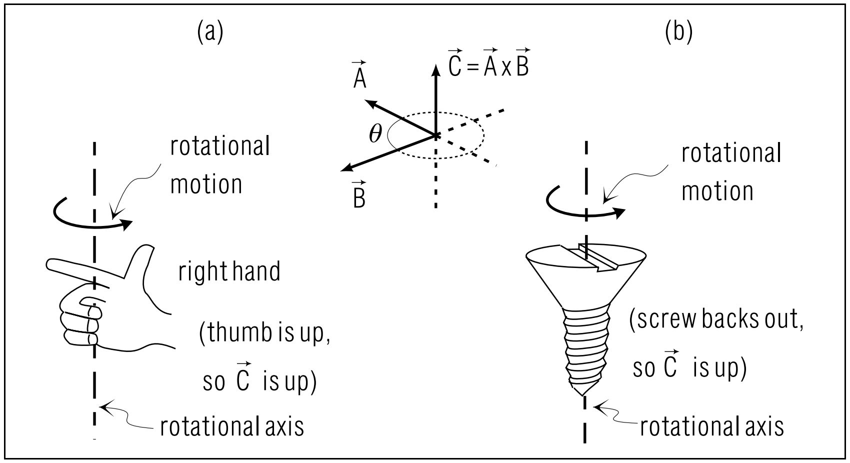
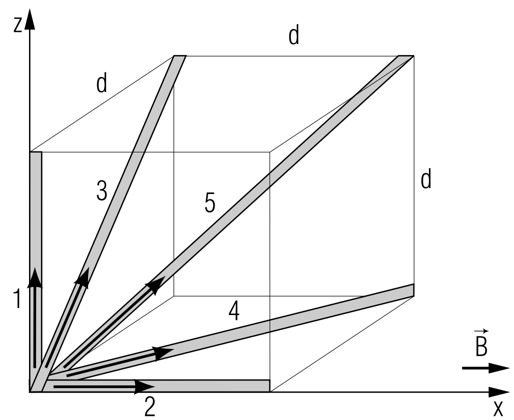


Figure 3. Two rules for finding the direction of $\vec{C} = \vec{A} \times \vec{B}$ by: (a) the "right hand" rule; (b) the "screw" rule.

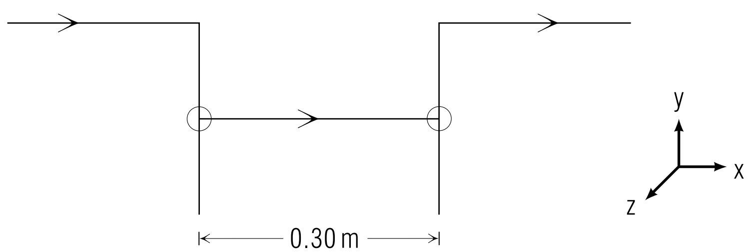
PROBLEM SUPPLEMENT

Note: Problems 6 and 7 also occur in this module's *Model Exam.*

1. Calculate the force on each of the five current-carrying wire segments shown below, if the field is $\vec{B} = 1.10\,\mathrm{T}\,\hat{x}$, $I = 3.0\,\mathrm{A}$, and $d = 0.15\,\mathrm{m}$. Consider each wire individually. *Help: [S-1]*

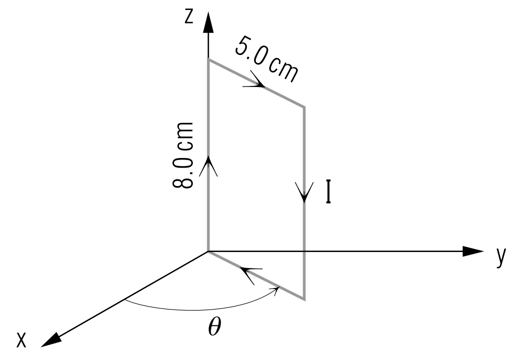


2. A section of a current-carrying wire is fixed so that it can slide up and down on two vertical metal guides as shown below. What magnetic field (magnitude and direction) is needed to prevent the sliding section from dropping and breaking the connection? The section is 0.30 m long, weighs 5 N, and has a current of 5.0 A passing through it. *Help: [S-5]*

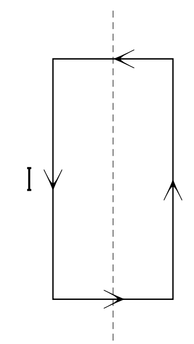


3. A rectangular current loop, 5.0 cm by 8.0 cm, is fixed on one side so that it rotates about the z-axis as shown below. If it carries a

current of 10 A and it is in a uniform magnetic field of $\vec{B} = 0.50\,\mathrm{T}\,\hat{x}$, calculate the torque about the z-axis acting on the loop when $\theta = 60^\circ$. *Help: [S-5]*

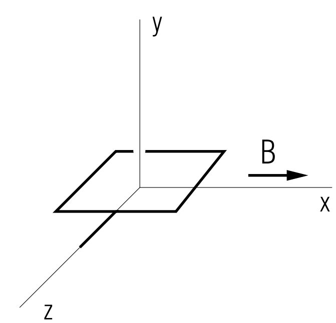


4. A rectangular current loop, 4.0 cm by 8.0 cm and carrying a current of 0.1 A, is suspended at a single point as shown. It is in a uniform horizontal magnetic field of magnitude 0.75 T. Find the magnitude of the torque acting on the loop when the normal to the plane of the loop makes an angle of 30° with respect to the magnetic field.

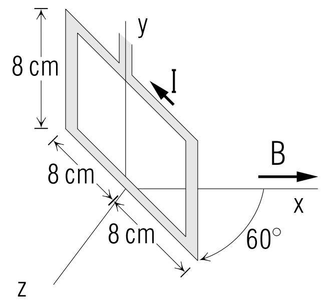


5. Calculate the magnitude of the maximum torque on a coil 3.0 cm by 5.0 cm, composed of 500 turns, when it carries a current of 1.0×10^{-3} A in a uniform magnetic field of magnitude 0.050 T. *Help: [S-7]*

6. A 500-turn square coil (2 cm on a side) in the x-z plane is in a magnetic field of magnitude $B = 0.16\,\mathrm{T}$ and direction $\hat{x}$. A current is passed through the coil, and it is observed that an external torque of $-1.6 \times 10^{-3}\,\mathrm{N\,m}\,\hat{z}$ is required to hold the coil in place. What are the magnitude and direction of I?



7. A rectangular current loop is in a magnetic field of magnitude $B = 0.5\,\mathrm{T}$ and direction $\hat{x}$. The plane of the loop makes a 60° angle with the direction of the magnetic field. The dimensions of the loop and direction of current are as shown in the figure, and $I = 20\,\mathrm{A}$. Calculate the forces on each of the four sides. Then calculate the torque, on the entire loop, about the y-axis.



Brief Answers:

1. $\vec{F}_1 = 0.495\,\mathrm{N}\,\hat{y}$; $\vec{F}_2 = 0$; $\vec{F}_3 = 0.495\,\mathrm{N}\,\hat{y} - 0.495\,\mathrm{N}\,\hat{z} = 0.495\,\mathrm{N}\,(\hat{y} - \hat{z})$; $\vec{F}_4 = -0.495\,\mathrm{N}\,\hat{z}$; $\vec{F}_5 = 0.495\,\mathrm{N}\,\hat{y} - 0.495\,\mathrm{N}\,\hat{z} = 0.495\,\mathrm{N}\,(\hat{y} - \hat{z})$.
2. Magnetic field must have a horizontal component of 3.3 T into the page. *Help: [S-2]*
3. $\vec{\tau} = -0.010\,\mathrm{N\,m}\,\hat{z}$
4. $\tau = 1.2 \times 10^{-4}\,\mathrm{N\,m}$
5. $\tau = 3.75 \times 10^{-5}\,\mathrm{N\,m}$
6. $I = 0.05\,\mathrm{A}$. Observed from above, I flows clockwise around the coil.
7. Top: $\vec{F} = -1.4\,\mathrm{N}\,\hat{y}$

   Front: $\vec{F} = -0.8\,\mathrm{N}\,\hat{z}$

   Bottom: $\vec{F} = 1.4\,\mathrm{N}\,\hat{y}$

   Back: $\vec{F} = 0.8\,\mathrm{N}\,\hat{z}$

   $\vec{t} = 6.4 \times 10^{-2}\,\mathrm{N\,m}\,\hat{y}$

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS, problem 1)*

1. Recall that we always use a right-handed coordinate system, i.e. $\hat{x} \times \hat{y} = \hat{z}$, so the positive y-direction is *into the page* (away from you). Thus the cube pictured in Problem 1 is in the quadrant of 3-dimensional space where all coordinates are positive. Do not be put off by the orientation of the axes in the figure: axes can be shown in any orientation as long as they show a right-handed coordinate system.
2. To begin the problem, do as always: Decide on the relevant equation to use, then write down each vector quantity in that equation in terms of unit vectors and given quantities, using symbols to represent the given quantities. One way to finish the problem is to then solve each component equation separately.

S-2 *(from PS, problem 2)*

The sliding section of current-carrying wire needs a magnetic force in the $+y$-direction to cancel the weight of the wire. The direction of the current (and thus $\vec{\ell}$) is in the $+x$-direction. Since $\vec{F}_B = I\vec{\ell} \times \vec{B}$ and $\vec{F}_B$ must be mutually perpendicular to $\vec{\ell}$ and $\vec{B}$, the only possible choice for $\vec{B}$ is in the positive or negative z-direction. Take either choice and apply the right-hand rule to determine if it gives the correct direction for $\vec{F}_B$.

S-3 *(from TX, 2a)*

The current in side a goes left so ℓ_a is in the $-\hat{x}$ direction. The magnetic field $\vec{B}$ goes right so $\hat{B}$ is in the $+\hat{x}$ direction. The angle between the two is 180° so their vector product is zero (see this module's *appendices*):

$$(-\hat{x}) \times (+\hat{x}) = 0\,.$$

S-4 *(from TX, 2a)*

The current in side b goes up so ℓ_b is in the $+\hat{y}$ direction. The magnetic field $\vec{B}$ goes right so $\hat{B}$ is in the $+\hat{x}$ direction. The angle between the two is 90° so their vector product is (see this module's *appendices*):

$$\vec{F}_B = ILB(\hat{y} \times \hat{x}) = ILB(-\hat{z})\,.$$

The direction $\hat{z}$ is given by (see this module's *appendices*): $\hat{x} \times \hat{y} = \hat{z}$, so the direction $+\hat{z}$ is *out of* the page and consequently $-\hat{z}$ is *into* the page.

S-5 *(from PS, Problems 1, 2, 3)*

First do the example in Sect. 2a, *in excruciating detail*. When you understand every nuance of that example, do the example in Sect. 2b in similar detail. Then you should be able to solve these problems on your own.

S-6 *(from TX, 2a)*

Recall that $\vec{r}_b$ is the vector from the rotation axis to the point of application of the force. Now notice that the force on side b is in the opposite direction to the force on side d (remember?). Then the axis about which the loop rotates cuts down through the *center* of the loop, bisecting sides a and c [see Fig. (1)]. Then the vector from that central vertical axis out to side b is: $\vec{r}_b = (W/2)\hat{x}$ [see Fig. (1)]. We already found that $\vec{F}_b = ILB(-\hat{z})$, so:

$$\begin{aligned}\vec{\tau}_b = \vec{r}_b \times \vec{F}_b &= (W/2)(ILB)\hat{x} \times (-\hat{z}) \\ &= -(W/2)(ILB)\hat{x} \times \hat{z} \\ &= +(W/2)(ILB)\hat{y}\end{aligned}$$

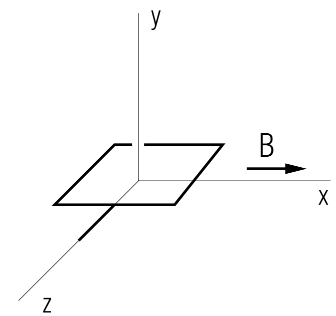
S-7 *(from PS, Problem 5)*

The word "500 turns" indicates that the continuous insulated wire was coiled around and around 500 times and then the whole set of 500 turns was glued together, keeping roughly the same shape as a single turn. The result is that 500 times as much current passes any one point on the loop as would pass that point if there was only one turn of wire.

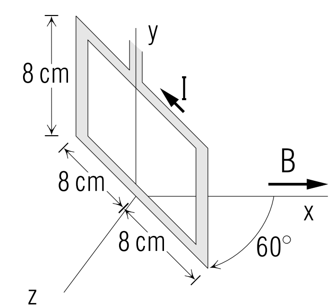
If you are having trouble finding the angle, look at the word *maximum*.

MODEL EXAM

1. See Output Skills K1-K2 in this module's *ID Sheet*.

2. 

A 500-turn square coil (2 cm on a side) in the x-z plane is in a magnetic field of magnitude $B = 0.16\,\mathrm{T}$ and direction $\hat{x}$. A current is passed through the coil, and it is observed that an external torque of $-1.6 \times 10^{-3}\,\mathrm{N\,m}\,\hat{z}$ is required to hold the coil in place. What are the magnitude and direction of I?

3. 

A rectangular current loop is in a magnetic field of magnitude $B = 0.5\,\mathrm{T}$ and direction $\hat{x}$. The plane of the loop makes a 60° angle with the direction of the magnetic field. The dimensions of the loop and direction of current are as shown in the figure, and $I = 20\,\mathrm{A}$. Calculate the forces on each of the four sides. Then calculate the torque, on the entire loop, about the y-axis.

Brief Answers:

1. See this module's *text*.
2. See Problem 6 in this module's *Problem Supplement*.
3. See Problem 7 in this module's *Problem Supplement*.