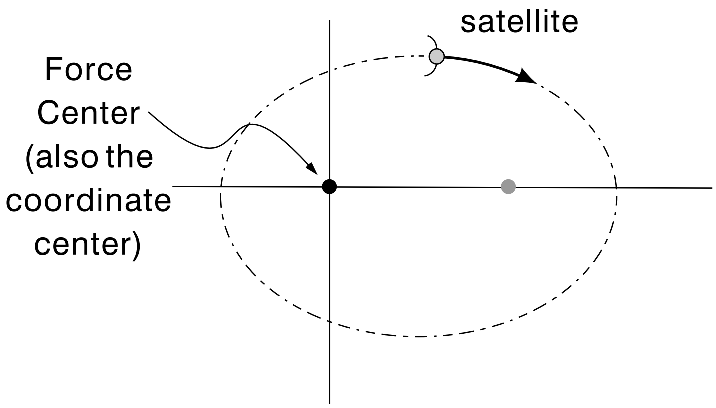
MISN-0-106

DERIVATION OF ORBITS IN INVERSE SQUARE LAW FORCE FIELDS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

DERIVATION OF ORBITS IN INVERSE SQUARE LAW FORCE FIELDS

by

Peter Signell

Title: **Derivation of Orbits in Inverse Square Law Force Fields**

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Length: 1 hr; 12 pages

Input Skills:

1. State the law of Conservation of Energy for a body acted on by a conservative force (MISN-0-21).
2. State the law of Conservation of Angular Momentum for planar motion (MISN-0-41).
3. Derive the gravitational potential energy function starting from the Universal Law of Gravitation (MISN-0-107).

Output Skills (Knowledge):

K1. Derive Kepler's first law, starting from Conservation of Angular Momentum, Conservation of Energy, and the potential energy function corresponding to the Universal Law of Gravitation.

Post-Options:

1. "Orbits in an Inverse Square Law Force Field: A Computer Project" (MISN-0-105).

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1. Introduction

The derivation of the elliptical orbits of the planets constitutes one of the greatest triumphs of Newtonian Mechanics. Perhaps you can imagine Newton's excitement when he completed his derivation and realized that for the first time in history the motions of the other planets and the earth would now be understood. In this unit the shape of orbits produced by the Law of Universal Gravitation is rigorously derived.

2. Derivation of Orbit Integral

Our notation is shown in Fig. 1. The position of the object is denoted by the polar coordinates r and θ, respectively, from the force-center and from the x-axis. Conservation of angular momentum for motion in this x-y plane gives:

$$L = mr^2 d\theta/dt = \text{ constant},$$

yielding:

$$d\theta = \frac{L}{mr^2} dt. \qquad (1)$$

Conservation of energy gives:

$$E = \frac{m(dr/dt)^2}{2} + \frac{L^2}{2I} - \frac{\gamma m\, m_E}{r} = \text{ constant},$$

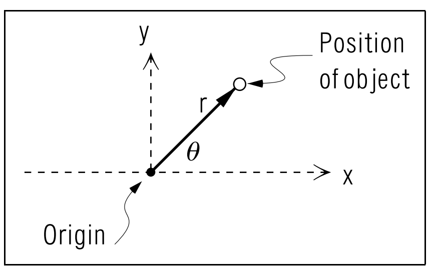
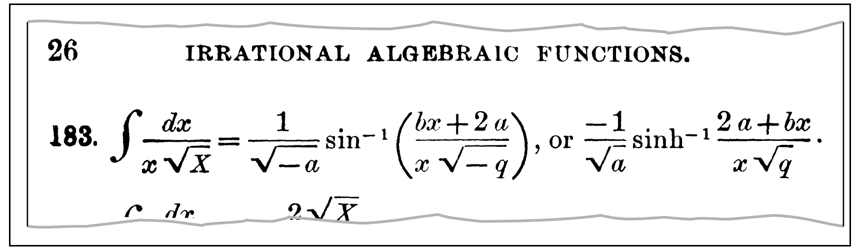


Figure 2. Polar coordinates.



26 IRRATIONAL ALGEBRAIC FUNCTIONS.

183. $\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bx+2a}{x\sqrt{-q}}\right)$, or $\frac{-1}{\sqrt{a}} \sinh^{-1} \frac{2a+bx}{x\sqrt{q}}$.

Figure 2. A fragment of a page from Pierce's Integrals (see text).

where we have added radial and angular kinetic energies to the gravitational potential energy.[1] We write the moment of inertia as $I = mr^2$ and collect the numerator factors of the gravitational potential energy into a single symbol:

$$\alpha \equiv \gamma m\, m_E.$$

Solving for radial velocity,

$$\frac{dr}{dt} = \sqrt{\frac{2E}{m} - \frac{L^2}{m^2r^2} + \frac{2\alpha}{mr}},$$

yielding:

$$dt = \frac{dr}{\sqrt{(2E/m) - (L^2/m^2r^2) + (2\alpha/mr)}}.$$

Substituting this into Eq. (1), we have:

$$d\theta = \frac{L\, dr}{mr^2\sqrt{(2E/m) - (L^2/m^2r^2) + (2\alpha/mr)}}.$$

We can integrate, with one integration constant, to find the connection between r and θ at various points on the orbit:

$$\theta - \theta_0 = \int \frac{L\, dr}{mr^2\sqrt{(2E/m) - (L^2/m^2r^2) + (2\alpha/mr)}}. \qquad (2)$$

3. Equation of the Orbit

There are several ways one can go about finding the integral in Eq. (2). One is to bring one power of r inside the square root:

$$\theta - \theta_0 = \frac{L}{m}\int \frac{dr}{r\sqrt{(2E/m)(r^2) - (L^2/m^2) + (2\alpha/m)(r)}}.$$

[1] See "Derivation of the Constants of the Motion for Central Forces" (MISN-0-58).

and look up this form in a table of integrals. For example, Fig. 2 contains a fragment of a page from Pierce.[2] Another way of solving the integral, not recommended as a general practice unless you have plenty of time available, is to put the integrand into a trivially-integrable form. In the present case, this can be effected by defining:

$$\beta \equiv \frac{L^2}{m\alpha} \qquad \text{and} \qquad \epsilon \equiv \sqrt{1 + (2E\beta/\alpha)}.$$

Then one can easily show that Eq. (2) is algebraically equivalent to:

$$\theta - \theta_0 = \int \frac{\beta\, dr}{\epsilon\, r^2 \sqrt{1 - \left(\frac{1}{\epsilon} - \frac{\beta}{\epsilon r}\right)^2}}.$$

Let $x \equiv (1/\epsilon) - (\beta/\epsilon r)$ so $dx = (\beta/\epsilon r^2)\, dr$ and the integral becomes:

$$\theta - \theta_0 = \int \frac{dx}{\sqrt{1 - x^2}}.$$

Finally, let $x = \sin y$ so:

$$dx = \cos y\, dy = \sqrt{1 - x^2}\, dy,$$

and:

$$dy = (dx)/\sqrt{1 - x^2}.$$

Then the final form of the integral is:

$$\theta - \theta_0 = \int dy = y = \sin^{-1} x = \sin^{-1}\left(\frac{1}{\epsilon} - \frac{\beta}{\epsilon r}\right),$$

or:

$$\sin(\theta - \theta_0) = \frac{1}{\epsilon} - \frac{\beta}{\epsilon r}.$$

In order to set θ_0 at a convenient value, we first solve for r:

$$r(\theta) = \frac{\beta}{1 - \epsilon \sin(\theta - \theta_0)}.$$

We will require that $\theta = 0^\circ$ gives the largest value of radius, an arbitrary but aesthetically pleasing requirement. Then:

$$r(0) = \frac{\beta}{1 + \epsilon \sin\theta_0},$$

[2] B. O. Peirce, *A Short Table of Integrals*, Ginn and Company, Boston (1929), form #183.

and it is obvious that we want $\theta_0 = -\pi/2$. Then:

$$r(\theta) = \frac{\beta}{1 - \epsilon \sin(\theta + \pi/2)},$$

or:

$$r(\theta) = \frac{\beta}{1 - \epsilon \cos\theta}. \tag{3}$$

This is the equation of the orbit.

4. Elliptical Orbits

To show that Eq. (3) corresponds to an ellipse requires that we show it can be put into the form:

$$\left(\frac{x - x_c}{a}\right)^2 + \left(\frac{y - y_c}{b}\right)^2 = 1,$$

where x_c and y_c are the coordinates of the center of the ellipse, and a and b are the semi-major and semi-minor axes. These quantities are illustrated in Fig. 3. It is obvious that, in our case, $y_c = 0$. The semi-major axis is easily calculated:

$$a = \frac{r(0) + r(\pi)}{2} = \frac{\beta}{1 - \epsilon^2}. \tag{4}$$

The semi-minor axis is the value of $(y = r\sin\theta)$ at the orbit point where $(x = r\cos\theta)$ is at the center of the ellipse. This is:

$$x_c = a - r(\pi) = \frac{r(0) - r(\pi)}{2} = \frac{\beta\epsilon}{1 - \epsilon^2}.$$

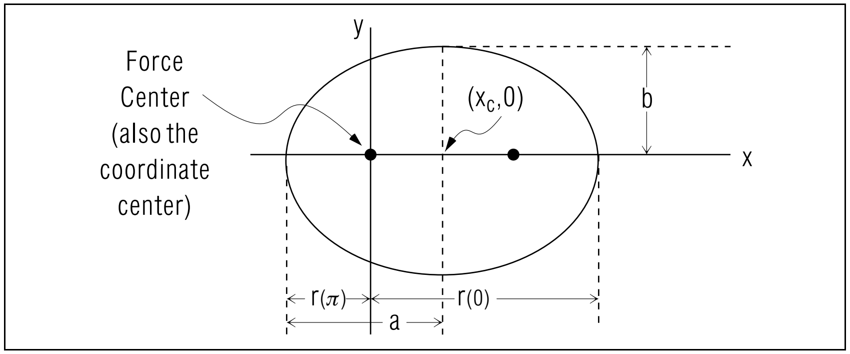


Figure 3. An elliptical orbit about a force center.

Now x_c is also given by:

$$x_c = r_c \cos\theta_c = \frac{\beta\cos\theta_c}{1-\epsilon\cos\theta_c},$$

so equating the two forms for x_c we get:

$$\frac{\beta\cos\theta_c}{1-\epsilon\cos\theta_c} = \frac{\beta\epsilon}{1-\epsilon^2},$$

and it is obvious by inspection that:

$$\cos\theta_c = \epsilon.$$

Then the semi-minor axis is:

$$b = r_c \sin\theta_c = \frac{\beta}{1-\epsilon\cos\theta_c}\sqrt{1-\cos^2\theta_c} = \frac{\beta}{\sqrt{1-\epsilon^2}}. \tag{5}$$

Thus to prove that $r(\theta)$ forms an ellipse, we must show that:

$$\left(\frac{x-x_c}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

where

$$x = r\cos\theta = \frac{\beta\cos\theta}{1-\epsilon\cos\theta}, \qquad y = r\sin\theta = \frac{\beta\sin\theta}{1-\epsilon\cos\theta},$$

and:

$$x_c = \frac{\beta\epsilon}{1-\epsilon^2}; \qquad a = \frac{\beta}{1-\epsilon^2}; \qquad b = \frac{\beta}{\sqrt{1-\epsilon^2}}.$$

This demonstration of algebraic manipulation is left to the reader to complete.[3] We should also note the alternative forms,

$$a = \frac{\beta}{1-\epsilon^2} = -\frac{\alpha}{2E} = \frac{\alpha}{2|E|},$$

and

$$b = \frac{\beta}{\sqrt{1-\epsilon^2}} = \frac{L}{\sqrt{2m|E|}}.$$

Note that E is negative because the mass m is bound to the force center: it can not escape unless it achieves positive total energy.[4] Also since E is negative, the eccentricity ϵ $(= \cos\theta_c)$ is between 0 and 1:

$$\epsilon \equiv \sqrt{1 + (2E\beta)/\alpha} = \sqrt{1 - (2|E|\beta)/\alpha} \le 1.$$

[3]An "elegant alternative" derivation is given in *Classical Mechanics*, V. Barger and M. Olsson, McGraw-Hill, New York (1973), p. 124-126.

[4]See "Gravitational Potential Energy" (MISN-0-107).

Acknowledgments

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MODEL EXAM

1. Derive Kepler's first law, starting from Conservation of Angular Momentum, Conservation of Energy, and the potential energy function corresponding to the Law of Universal Gravitation. Use: $\beta \equiv L^2/(m\alpha)$, $\epsilon \equiv \sqrt{1 + (2E\beta/\alpha)}$, $x \equiv (1/\epsilon) - (\beta/\epsilon r)$.

From B. O. Peirce's *A Short Table of Integrals*, Ginn and Company (1929), with $X = a + bx + cx^2$ and $q = 4ac - b^2$:

$$\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}} \log\left(\sqrt{X} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right).$$

$$\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}} \sinh^{-1}\left(\frac{2cx + b}{\sqrt{q}}\right).$$

$$\int \frac{dx}{\sqrt{X}} = \frac{-1}{\sqrt{-c}} \sin^{-1}\left(\frac{2cx + b}{\sqrt{-q}}\right).$$

$$\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bx + 2a}{x\sqrt{-q}}\right).$$

$$\int \frac{dx}{x\sqrt{X}} = \frac{-1}{\sqrt{a}} \sinh^{-1}\left(\frac{2a + bx}{x\sqrt{q}}\right).$$

$$\int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}, \text{ if } a = 0.$$

$$\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}.$$