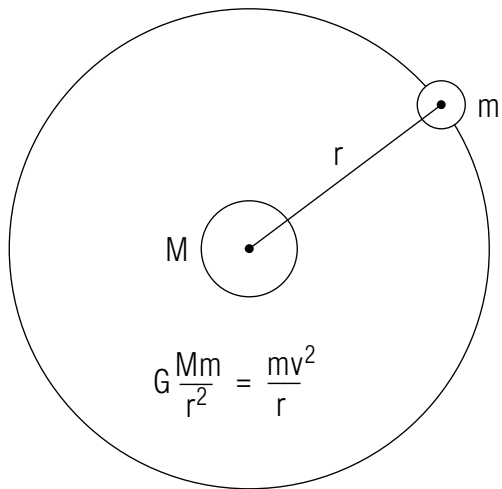


ORBITAL MOTION IN AN INVERSE-SQUARE-LAW FORCE FIELD



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by
Paul M. Parker

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Input Skills:

1. Vocabulary: Newton's law of universal gravitation, universal gravitational constant (MISN-0-101); period, angular velocity (MISN-0-9); angular momentum (MISN-0-41); conic section, eccentricity, (glossary).
2. Solve problems involving uniform circular motion produced by a given centripetal force (MISN-0-17).
3. Calculate the gravitational force vector on one mass due to a given configuration of other masses (MISN-0-101).

Output Skills (Knowledge):

- K1. Vocabulary: apogee, central force, perigee.
- K2. State Kepler's three kinematical laws of planetary motion.
- K3. State how Newton's law of universal gravitation may be used to justify Kepler's laws.

Output Skills (Problem Solving):

- S1. Given a table of solar system data, use Newton's law of universal gravitation and the value of g at the earth's surface to determine the period of revolution, orbital speed, and centripetal acceleration of a given circularly orbiting satellite.

Post-Options:

1. "Derivation of Newton's Law of Gravitation" (MISN-0-103).
2. "Orbits in an Inverse-Square-Law Force Field: A Computer Project" (MISN-0-105).
3. "The Equivalence Principle: An Introduction to Relativistic Gravitation" (MISN-0-110).

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1. Historical Overview

1a. Introduction. Since earliest recorded human history, the motion of the planets in the heavens has attracted the attentions of the curious. Against the background of the stars whose positions in the sky are fixed relative to each other, the motions of the planets appear quite complicated and much attention was given over the centuries to arriving at some sort of explanation for this seemingly mysterious behavior of the planets.

1b. Copernicus Proposes a Heliocentric Solar System. It was the Polish astronomer Nicolaus Copernicus (1473-1543) who advanced the idea that all the then known planets were in orbit around a single star, our sun. In this heliocentric view, the earth is merely one of a number of sun-orbiting planets, not the center of the universe as it was in the then widely accepted view of Claudius Ptolemy of Alexandria (2nd century AD) and his successors. Copernicus discovered that from the heliocentric point of view, the planets appear to orbit the sun smoothly and without the puzzling intricacies that earth-bound observation introduces.

1c. Astronomical Observations Lead to Laws of Motion. The Danish astronomer Tycho Brahe (1546-1601) spent most of his adult life recording the motions of the planets with unprecedented accuracy. His German disciple and colleague Johannes Kepler (1571-1630) continued his work but unlike Brahe, Kepler enthusiastically accepted the heliocentric theory of Copernicus. This eventually allowed him to extract from the massive and accurate data available to him a set of three laws of planetary motion of general validity.

2. Kepler's Laws of Planetary Motion

2a. Kepler's First and Second Laws. Kepler's first law states that all planets travel in elliptical paths with the sun at one of the two foci of the ellipse.¹ The second law states that the radius vector \vec{r} between

¹The data indicated that those ellipses were nearly circular, but true ellipses they were.

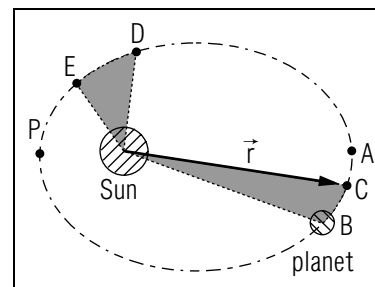


Figure 1. An illustration of Kepler's second law.

the sun and the instantaneous position of a planet “sweeps out” equal amounts of area of ellipse in equal intervals of time. This means that if the time it takes the planet to go from B to C in Fig. 1 is equal to the time it takes the planet to go from D to E , the shaded areas in Fig. 1 will be equal. As a consequence, the further the planet is from the sun, the slower it will move. The planet moves fastest at the perigee P (the point of closest approach) and slowest at the apogee A (the point farthest away from the sun). For the limiting case of a circle Kepler's second law requires the orbiting speed to be constant and the type of motion therefore is uniform circular motion.

2b. Kepler's Third Law. While the first and second laws apply to each planet individually, the third law relates the motions of the planets to each other. The law states that the periods T_1 and T_2 of two planets are related to the mean radii R_1 and R_2 of their respective orbits by

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}. \quad (1)$$

The period of an orbital motion is defined as the time required to complete one orbit. The mean radius of an elliptic orbit is defined as the semi-major axis of the ellipse, one half the distance between apogee and perigee. The discovery of the third law, Kepler writes, was the culmination of seventeen years of intensive work. “At last, the true relation ... overcame by storm the shadows of my mind ... with such fullness of agreement ... that I at first believed that I was dreaming.”

3. Newton's Interpretation

3a. Newton Applied His Law of Gravitation. Newton's discovery of the law of universal gravitation,

$$F = G \frac{m_1 m_2}{r^2}, \quad (2)$$

naturally led him to investigate whether this type of force, when used in the second law of mechanics, $\vec{F} = m\vec{a}$, would provide the dynamical explanation of the three kinematical laws of Kepler. Newton found that indeed it did.

3b. Kepler's Second Law Results From a Central Force. The second law of Kepler is a consequence of the fact that the gravitational force exerted by the sun and acting on a planet is a central force, i.e., at all times directed along the radius vector and pointing toward the sun, and not having at any time a component, however small, perpendicular to the radius. A force of this nature cannot change the angular momentum of the planet, mvr , around the sun. Therefore, conservation of angular momentum requires that any decrease in v is compensated for by an increase in r , and vice versa. This can be shown to require the exact behavior specified by Kepler's second law.

3c. Inverse-Square Force Yields Kepler's First & Third Laws.

The first and the third laws require the force to be specifically an inverse-square force, that is, its magnitude must vary as $1/r^2$. The detailed proofs are beyond the scope of this module, but a proof of the third law for circular orbits will be given in Section 4. It can be shown that a compact object moving under the influence of a central, inverse-square force will move on a path which is a conic section: a hyperbolic path if the force is repulsive, and a circular, elliptic or parabolic path if the force is attractive. Since the gravitational force is attractive, closed orbits must therefore be elliptic (with the circular orbit as a special case). Open orbits are parabolic, such as those of non-returning comets which pass through the solar system only once. (Halley's comet is a returning comet

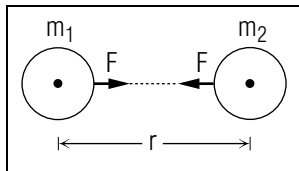


Figure 2. The gravitational attraction of two masses.

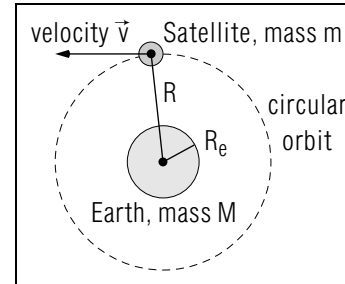


Figure 3. A satellite orbiting the earth.

which moves on a highly eccentric² elliptic path.) Thus, Kepler's first law is a direct consequence of the gravitational force being central and inverse-square.

4. Circular-Orbit Satellite Motion

4a. Overview. In this section, we will examine the motion of a satellite orbiting a spherically symmetric, massive object under the influence of the universal gravitational force. This could be the motion of a spaceship orbiting the earth, or a spaceship orbiting another planet, moon, the sun or another star. It could also be the motion of a moon orbiting its planet, or the motion of a planet in (near) circular motion around the sun. We will discuss this type of motion in terms of a spaceship orbiting the earth but the discussion applies equally well to any other of the above situations.

4b. Gravity Provides The Centripetal Force. Let the satellite move with uniform circular motion on a path of radius R , and therefore a distance of $(R - R_e)$ above the surface of the earth. The centripetal force acting on the satellite is the universal gravitational force exerted by the earth and its magnitude must equal the satellite's mass m times its centripetal acceleration v^2/R :

$$G \frac{Mm}{R^2} = \frac{mv^2}{R}. \quad (3)$$

The mass of the satellite m can be cancelled which means that Eq. (3) is valid for a satellite of any mass. Solving for v one obtains

$$v = \left(\frac{GM}{R} \right)^{1/2}. \quad (4)$$

²The words "highly eccentric" mean that the orbit is far from being circular.

The period (time of one revolution) for uniform circular motion is

$$T = \frac{\text{distance for one revolution}}{\text{speed during the revolution}} = \frac{2\pi R}{v}, \quad (5)$$

and using the expression for v from Eq. (4) gives

$$T = 2\pi \left(\frac{R^3}{GM} \right)^{1/2}. \quad (6)$$

This shows that the square of the period of the satellite motion is proportional to the cube of the radius R of the path, in agreement with Kepler's third law, at least for circular orbits. We can define an "acceleration due to gravity at radius R from the center of the earth," g_R , and write (where $R > R_e$):

$$g_R = \frac{GM}{R^2}, \quad (7)$$

This allows the expressions for v and T to be expressed in the more compact forms

$$v = (g_R R)^{1/2}; \quad T = 2\pi \left(\frac{R}{g_R} \right)^{1/2}, \quad (8)$$

For a tightly orbiting satellite, say, a spaceship only a hundred miles above earth, $R \approx R_e = 6.37 \times 10^6$ m, $g_R \approx g = 9.81$ m/s², from which one determines a period T of roughly ninety minutes and an orbital speed of about 8 km/s. Note that for a given R , both v and T of the satellite's motion are fixed. As can be seen most directly from Eqs. (4) and (6), the larger the orbit, the longer the period and the slower the speed.

4c. The Satellite Travels in Free-Fall. The spaceship and its passengers at all times experience an acceleration equal to the local acceleration of gravity. They are therefore at all times in a free-fall condition which is a condition equivalent to weightlessness. Note, then, that the spaceship is weightless not because it is in some gravity-free space, which it isn't, but because it is accelerating freely in the very real gravitational field of the earth.

5. Conclusion

5a. Planets May Perturb the Orbits of Neighbors. Extremely refined calculations of planetary motions must take into account the secondary effects of the gravitational attraction of the planets on each other.

Careful analysis of these small, so-called perturbations clears up remaining discrepancies and allows the prediction of the positions of the planets over the centuries with split-second accuracy. Such calculations led to the prediction of the existence of two heretofore unknown planets and their subsequent discovery, the planet Neptune in 1846 and the planet Pluto in 1930.

5b. Gravitational Mass vs. Inertial Mass. Some reflection shows that mass, traditionally defined as the amount of matter contained in a given object, appears in physics in two different aspects: (1) as the coefficient of inertia in Newton's second law - the more massive an object, the larger a force is required to produce a specified amount of acceleration; and (2) as gravitational mass: the more massive an object, the larger a gravitational force it exerts and experiences. There is no logical necessity that the mass of an object when determined by inertial methods is, in the same units, equal to the mass of the object when measured by gravitational methods (for example, by weighing it). It is therefore meaningful to ask whether the inertial mass of an object differs from its gravitational mass, however small the difference may be. Experiments to answer this question were first carried out by the Hungarian physicist Eötvös in 1909 and continued by others. All these experiments have so far shown that there is no difference between inertial and gravitational mass to the accuracy of the experiments. The best accuracy on record is about one part in 10^{11} . In other words, if there is a difference between the gravitational and inertial mass of an object, it will not show up before the eleventh digit in the measurement results! Therefore, the distinction between inertial and gravitational mass is superfluous and no such distinction is required. The equivalence between gravitational and inertial mass has been exploited by Einstein in 1911 and forms the cornerstone of his general theory of relativity which is primarily a theory of gravitation, constituting the first significant advance since the days of Newton.

Acknowledgments

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Glossary

- **apogee:** the point of a satellite's orbit furthest from the focus of its elliptical path.
- **central force:** a force which is radial along the line joining the two interacting systems.
- **perigee:** the point of a satellite's orbit closest to the focus of its elliptical path.
- **conic section:** a plane curve which may be obtained by cutting a cross section of a cone. The possibilities include a circle, an ellipse, a parabola, and a hyperbola.
- **eccentricity:** the ratio of the distance from the focus to a point on a conic section, to the distance from a fixed straight line to the same point on the conic section. The numerical value of the eccentricity, e , determines what kind of curve the conic section is:

$e > 1 \implies$ hyperbola

$e = 1 \implies$ parabola

$e < 1 \implies$ ellipse

$e = 0 \implies$ circle

PROBLEM SUPPLEMENT

Note: Problem 7 also occurs in this module's *Model Exam*.

1 lb (force) = 4.448 N; 1 km = 0.6214 mi; 1 yr = 3.167×10^7 s

PHYSICAL AND ORBITAL DATA FOR THE SOLAR SYSTEM*				
Body	Mean Radius ^c (meters)	Mass (kg)	Mean Orbital Radius (meters)	Orbital Period (seconds)
Sun	6.96×10^8	1.99×10^{30}	-----	-----
Mercury ^a	2.43×10^6	3.30×10^{23}	5.79×10^{10}	7.60×10^6
Venus ^a	6.06×10^6	4.87×10^{24}	1.08×10^{11}	1.94×10^7
Earth	6.37×10^6	5.98×10^{24}	1.50×10^{11}	3.16×10^7
Mars ^a	3.37×10^6	6.40×10^{23}	2.28×10^{11}	5.94×10^7
Jupiter	6.99×10^7	1.90×10^{27}	7.78×10^{11}	3.74×10^8
Saturn	5.84×10^7	5.69×10^{26}	1.43×10^{12}	9.30×10^8
Uranus	2.30×10^7	8.73×10^{25}	2.87×10^{12}	2.65×10^9
Neptune	2.22×10^7	1.03×10^{26}	4.50×10^{12}	5.20×10^9
Pluto	$< 3 \times 10^6$	$< 6 \times 10^{23}$	5.90×10^{12}	7.82×10^9
Moon ^b	1.74×10^6	7.35×10^{22}	3.84×10^8	2.36×10^6

*Data adapted from the Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac, Her Majesty's Stationery Office, London (1961). Notes: ^aRadius obtained from radar observations, mass from space probe perturbations. ^bOrbital data are with respect to the earth. ^cRadius of a sphere of equal volume.

1. Phobos, a moon of Mars, travels in a circular orbit of radius 9.4×10^6 m about that planet. Starting from the law of universal gravitation, calculate Phobos's:
 - a. orbital speed
 - b. period of revolution
 - c. centripetal acceleration
2. Using the law of universal gravitation and assuming circular orbits with the heavier body fixed at center of the circle, calculate the period of revolution (in days) of:

- a. the earth about the sun
- b. the moon about the earth.

Compare your values to those in the table, which are: (a) a year; and (b) a lunar month.

3. Find the orbital speed of a satellite in orbit 200 km above the earth.
4. Calculate the shortest possible period of revolution for a spaceship orbiting:
 - a. the earth
 - b. the moon.
5. What is the shortest period of revolution for a satellite revolving about a planet whose radius is 1.0×10^3 km and whose surface value of g is 5.0 m/s^2 ? Calculate the orbital speed and centripetal acceleration of a satellite 1.0×10^2 km above the planet's surface.
6. Find the distance above the earth's surface (in earth radii) a satellite orbiting in the earth's equatorial plane³ must be in order for it to be stationary above a certain location at all times. (If the orbit does not lie in the equatorial plane, the satellite will still be in the same location every day at a given time of day - a very useful property for a communications satellite.)
7. A spacecraft is in a circular orbit about the moon, 8.0×10^4 miles above the moon's surface. Starting from Newton's law of universal gravitation, calculate the spacecraft's:
 - a. orbital speed,
 - b. period of revolution,
 - c. centripetal acceleration.

Do not use a remembered value for the universal gravitational constant G . Instead, use your instant-recall value of g at the earth's surface and the radius of the earth (in the Solar System Data Table) to find G .
Help: [S-4]

³The earth's equatorial plane is the plane that includes the earth's equator.

Brief Answers:

1. a. $v = 2.13 \times 10^3 \text{ m/s}$.
 b. $T = 2.77 \times 10^4 \text{ s} = 7.69 \text{ hr}$
 c. $a = 0.483 \text{ m/s}^2$, radially inward.
2. a. $T = 3.17 \times 10^7 \text{ s} = 367 \text{ days}$ ($1 \text{ yr.} = 3.16 \times 10^7 \text{ s}$).
 b. $T = 2.37 \times 10^6 \text{ s} = 27.4 \text{ days}$ (lunar month = $2.36 \times 10^6 \text{ s}$).
3. $7.79 \times 10^3 \text{ m/s}$.
4. a. $T = 5.06 \times 10^3 \text{ s} = 84.3 \text{ min.}$ *Help: [S-1]⁴*
 b. $T = 6.52 \times 10^3 \text{ s} = 109 \text{ min.}$ *Help: [S-1]*
5. $T = 2.8 \times 10^3 \text{ s}$,
 $v = 2.1 \times 10^3 \text{ m/s}$, *Help: [S-3]*
 $a = 4.1 \text{ m/s}^2$. *Help: [S-3]*
6. 5.6 radii above the earth's surface. *Help: [S-6]*
7. a. $1.62 \times 10^3 \text{ m/s}$. *Help: [S-5]*
 b. $7.25 \times 10^3 \text{ s}$. *Help: [S-5]*
 c. $-(1.40 \text{ m/s}^2) \hat{r}$. *Help: [S-5]*

⁴The symbol [S-1] means that if you have trouble getting the answer you can get help from sequence [S-1] in this module's *Special Assistance Supplement*.

SPECIAL ASSISTANCE SUPPLEMENT

S-1 (from PS, problem 4)

Remember the equations that make up Eq. (3), use them to derive Eq. (6), note that minimum T means minimum R . What is the minimum R for a satellite orbiting the earth? For one orbiting the moon?
Help: [S-2]

S-2 (from [S-1])

It would make no sense to design an orbit that is below ground level!

S-3 (from PS, problem 5)

$R_p = 1.0 \times 10^6 \text{ m}; \quad R = 1.1 \times 10^6 \text{ m}.$
 $g = 5.0 \text{ m/s}^2.$

$$g_R = \left(\frac{R_p}{R}\right)^2 g = 4.1 \text{ m/s}^2.$$

$$v_R = \sqrt{g_R R} = 2.1 \times 10^3 \text{ m/s}.$$

S-4 (from PS, problem 7)

The derivation of G from g and R_e is shown in this module's *text*.

S-5 (from PS, problem 7)

Orbital speed, orbital period, and centripetal acceleration are all covered adequately (for this problem) in this module's *text*. For a more elementary discussion of these quantities, see the Index.

S-6 (from PS, problem 6)

What is the orbital period of all points on the surface of the earth? Then what is the orbital period of the satellite if it stays above a point on the surface of the earth?

S-7 (from [S-5])

A question: How far is the satellite from the center of the moon?

2. See this module's *Problem Supplement*, problem 7.

MODEL EXAM

PHYSICAL AND ORBITAL DATA FOR THE SOLAR SYSTEM*				
Body	Mean Radius ^c (meters)	Mass (kg)	Mean Orbital Radius (meters)	Orbital Period (seconds)
Sun	6.96×10^8	1.99×10^{30}	-----	-----
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Moon ^b	1.74×10^6	7.35×10^{22}	3.84×10^8	2.36×10^6

*Data adapted from the Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac, Her Majesty's Stationery Office, London (1961). Notes: ^aRadius obtained from radar observations, mass from space probe perturbations. ^bOrbital data are with respect to the earth. ^cRadius of a sphere of equal volume.

1. See Output Skills K1-K3 in this module's *ID Sheet*. One or more of these skills, or none, may be on the actual exam.
2. A spacecraft is in a circular orbit about the moon, 8.0×10^4 miles above the moon's surface. Starting from Newton's law of universal gravitation, calculate the spacecraft's: (a.) orbital speed; (b) period of revolution; (c) centripetal acceleration.

Do not use a remembered value for the universal gravitational constant G . Instead, use your instant-recall value of g at the earth's surface and the radius of the earth to find G .

Brief Answers:

1. See this module's *text*.

