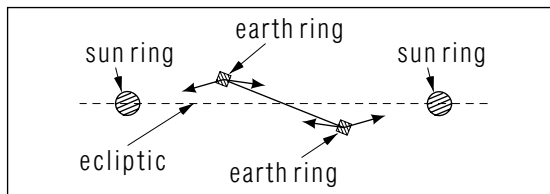
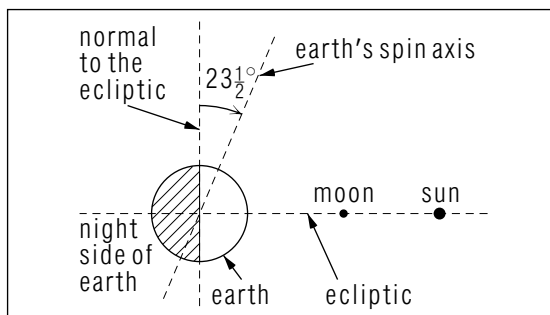


ANGULAR MOMENTUM AND TORQUE: PRECESSION



ANGULAR MOMENTUM AND TORQUE: PRECESSION

by

Peter Signell and J. S. Kovacs, Michigan State University

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Author: Peter Signell and J. S. Kovacs, Michigan State University

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Input Skills:

1. Define angular momentum for rigid body rotation about a fixed axis (MISN-0-41).
2. State the relationship between torque and change in angular momentum for rigid body rotation about a fixed axis (MISN-0-33).
3. Calculate vector cross products (MISN-0-2).
4. Calculate vector torques (MISN-0-5).
5. Use Newton's Second Law in appropriate simple problems (MISN-0-16).

Output Skills (Knowledge):

- K1. Describe precession and its origin, including identification of the axis about which precession occurs.
- K2. Derive the frequency of precession of a top in terms of its center-of-gravity radius, its mass, and its spin angular momentum.
- K3. Describe and explain the Precession of the Equinoxes.

Output Skills (Problem Solving):

- S1. Work problems involving the simple precession of objects such as tops and gyroscopes.

Post-Options:

1. "Stability of a Bicycle" MISN-0-38).
2. "Rotational Motion of a Rigid Body" (MISN-0-36).

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ANGULAR MOMENTUM AND TORQUE: PRECESSION

by

**Peter Signell and J. S. Kovacs, Michigan State
University**

1. Vectors: Succinct, Graphic

The Law of Conservation of Angular Momentum is often introduced with simple illustrations involving circular motions of rigid objects about fixed axes of rotation, and this is sufficient for understanding a number of natural and man-made phenomena. For such illustrations, you need not treat angular momentum, torque, and angular velocity as explicit vectors. However, you must then specify “the torque about the axis,” “the angular momentum about the axis,” or “the angular velocity about the axis.” Such statements make it sound like one is talking about components of vectors (i.e. components along such axes) and indeed that is the case. A switch to vector notation will then allow you to stop having to always add the phrase “about the axis.” You will also then be able to picture phenomena like precession very graphically and will be prepared to move further toward the ability to understand and control general types of motions.

2. Angular Momentum Components

2a. The Law of Conservation of Angular Momentum. The angular momentum of an object in motion about an axis is conserved if the object experiences no torque about that axis.

2b. Calculation of Angular Momentum. In order to use Conservation of Angular Momentum, one must be able to compute it for any given situation. For the case of circular motion of a point object with mass m , orbital radius r , and speed v , the angular momentum about the rotational axis is:

$$L = mvr \text{ (point mass, circular motion).}$$

For a number of rigidly-connected point masses in circular motion about an axis the individual-mass angular momenta add to give the total angular

momentum about that axis:

$$L = \sum_i L_i = \sum_i m_i v_i r_i = \sum_i m_i r_i^2 \omega_i.$$

Using the fact that all points on a rigid object have identical angular velocities,¹ we can drop the subscript on ω and factor it out:

$$L = \omega \sum_i m_i r_i^2 \equiv I\omega \quad (1)$$

where I is the moment of inertia about the rotational axis.

2c. Derivation of Conservation of Angular Momentum. For the circular motion of rigid objects, one can easily derive Conservation of Angular Momentum from Eq. (1) by differentiating it with respect to time:

$$\frac{dL}{dt} = I\alpha \quad (2)$$

where α is the object’s angular acceleration. The right hand side of Eq. (2), however, is equal to the total torque along the rotational axis:² $dL/dt = \tau$. If that torque is zero then the rate of change of L along the axis is also zero and the value of L along the axis must be constant (conserved).

3. Vector Angular Momentum

3a. Definition of Vector Angular Momentum. The vector equation

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

can easily be derived from Newton’s Second Law if one defines a point object’s vector angular momentum about any reference point to be

$$\vec{L} = \vec{r} \times \vec{p} \quad (3)$$

where \vec{r} is the radius vector from the reference point to the object and \vec{p} is the object’s momentum.

¹See “Uniform Circular Motion: Kinematics, Moment of Inertia, Conservation of Angular Momentum” (MISN-0-9).

²See “Torque and Angular Acceleration” (MISN-0-33) for a derivation of $\tau = I\alpha$.

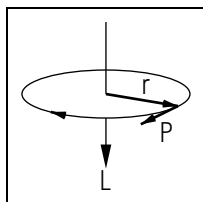


Figure 1. The angular momentum vector of a spinning disc.

3b. Derivation of $\vec{\tau} = d\vec{L}/dt$. Differentiating Eq. (3) with respect to time and using the chain rule for differentiation gives:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt} \vec{r} \times \vec{p} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \quad (4)$$

The cross-product in the first term on the right hand side of Eqnref4 is zero.³ The remainder of Eqnref4 becomes, using Newton's second law⁴ and the definition of torque:⁵

$$\frac{d\vec{L}}{dt} = \vec{\tau} \quad (5)$$

3c. Conservation of \vec{L} . If the vector torque is zero, then Eq. (5) says the time rate of change of \vec{L} is zero and hence \vec{L} must be a constant vector. This means that neither its magnitude nor direction can change with time.

3d. Examples: Frisbees, Skipping Stones. A rotating disc is a good example of a use of conservation of (vector) angular momentum. In the disc in Fig. 1 the rotation is clockwise as seen from above. Then the rule of Eq. (4) produces a downward pointing angular momentum vector for each element of mass in the disc. Adding these individual mass contributions, the total angular momentum must also be downward.

$$\sum_i \vec{L}_i = \vec{L} \text{ total (vector) angular momentum}$$

No matter how wobbly the manner in which one starts the disc spinning, its resulting angular momentum vector will not wobble but will remain

³See "Mathematical Skills: Addition, Subtraction and Products of Vectors" (MISN-0-2) for proof that the cross product of a vector with itself is zero.

⁴See "Free-Body Force Diagrams, Frictional Forces, Newton's Second Law" (MISN-0-15) for an explanation of Newton's Second Law.

⁵See "Force and Torque" (MISN-0-5) for an explanation of torque.

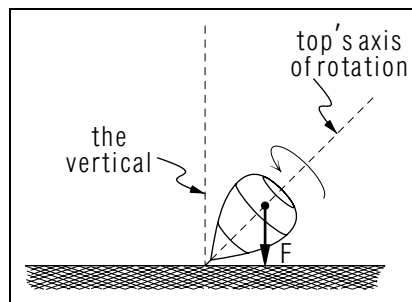


Figure 2. A spinning top and the earth's gravitational force on it.

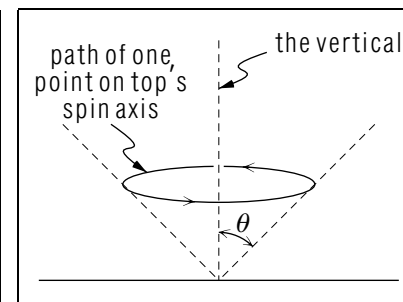


Figure 3. The top's spin axis traces out a conical surface.

fixed in magnitude and direction once external torques are removed. Thus the stability of orientation of a spinning disc, such as a Frisbee or a skipping stone.

4. Precession

4a. Description of Precession. One of the most interesting predictions of the vector torque-angular momentum relation, $\vec{\tau} = d\vec{L}/dt$, is the phenomenon of *precession*. This is defined as the cone-shaped motion of a rotational axis due to the action of a constant torque on a spinning object. Such motion is commonly seen in gyroscopes and tops, and is illustrated for the latter in Fig. 2. The top's spin axis moves as though along the surface of a cone: the cone's axis is along the direction of the gravitational force. This is shown in Fig. 3, where the cone's half-opening angle is denoted θ .

Such motion is defined as precession. Note that the direction of motion of any point on the spin axis is always at right angles to the

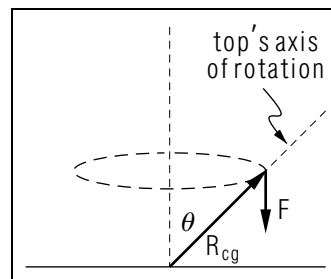


Figure 4. The torque on the top is at right angles to the gravitational force and the vector R to the top's center of gravity.

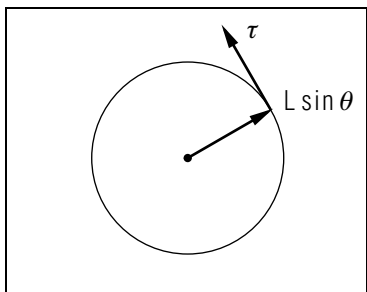


Figure 5. Top view of a top's $\vec{\tau}$ and \vec{L} at a particular instant.

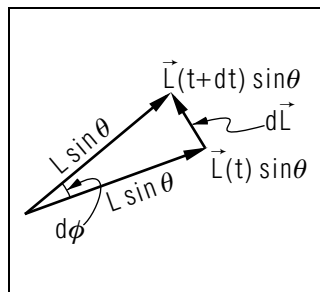


Figure 6. As in Fig. 5, but showing also \vec{L} after it has moved slightly in the direction of $\vec{\tau}$.

precession axis and hence to the force direction. You can use that fact to quickly sketch the precessional trajectory of any given point on the spin axis, as in Fig. 3.

4b. Conservation of $|\vec{L}|$. The conical motion of a top's rotational axis can be easily explained by manipulation of the vector torque-angular momentum equation, $\vec{\tau} = d\vec{L}/dt$. As illustrated in Figs. 2 and 4, the torque produced by the gravitational force of the earth is always at right angles to the top's angular momentum vector which is taken here to be along its rotational axis. This means that the component of torque along the direction of the angular momentum vector is zero and hence the magnitude of the angular momentum *is* conserved.⁶

4c. Precession of \hat{L} . The torque on a simple spinning top due to the earth's gravitational field is always at right angles to the top's spin angular momentum vector, \vec{L} , and hence by $\vec{\tau} = d\vec{L}/dt$ only the direction of \vec{L} , \hat{L} , can change. This is a good assumption if there is no observable nutation and if the top's angular momentum due to precessional motion is small compared to its spin angular momentum.

In order to calculate the rate with which the direction of \vec{L} changes, it is convenient to view the torque and \vec{L} vectors from above as in Fig. 5. In that figure θ is the angle the top's angular momentum vector makes with the vertical (See Fig. 3.). Note in that \vec{L} 's direction must change as indicated by $\vec{\tau}$. The increment of change $d\vec{L}$ produced in time dt will

⁶Simply take components of $[\vec{\tau} = d\vec{L}/dt]$ along the direction of angular momentum to obtain $[0 = dL/dt]$.

amount to a rotation of \vec{L} through an angle $d\phi$ (to be determined) as in Fig. 6.

Using an infinitesimal time, the length of $d\vec{L}$ can be equated to the arc of radius $(Ld\phi \sin \theta)$ between $\vec{L}(t)$ and $\vec{L}(t + dt)$:

$$d\vec{L} = L \sin \theta d\phi \hat{\tau}$$

hence

$$\vec{\tau} = \frac{d\vec{L}}{dt} = L \sin \theta \frac{d\phi}{dt} \hat{\tau} \equiv L \sin \theta \dot{\phi} \hat{\tau}$$

and the angular velocity of precession about the vertical is:⁷

$$\dot{\phi} = \tau / (L \sin \theta) \quad (6)$$

Referring back to Fig. 4, it is easily seen that the torque $(\vec{r} \times \vec{F})$ is:

$$\tau = R_{cg} mg \sin \theta ,$$

and so, from Eq. (6),

$$\dot{\phi} = \frac{R_{cg} mg}{L} . \quad (7)$$

For an examination of nutation, a looping motion which can occur during precession, we refer you to other references.⁸

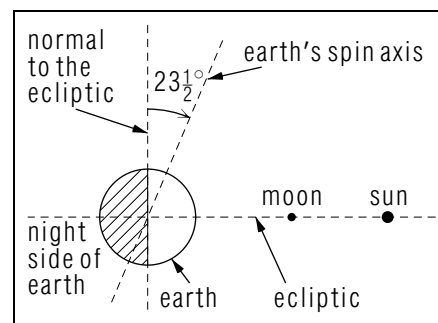


Figure 7. Sketch showing the 23.5° tilt of the earth's axis (not to scale) with respect to the perpendicular to the ecliptic plane.

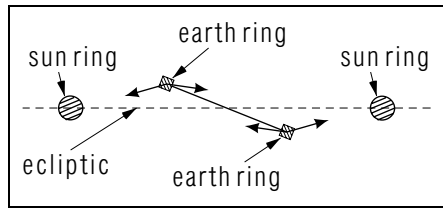


Figure 8. Cross section of the effective sun mass ring and earth equatorial ring.

5. Precession of the Equinoxes

5a. Observations of the Precession. If one observes the positions of the constellations in the night sky at the same time each year,⁹ each year, these positions will be observed to rotate about the sky taking 25,800 years for one complete revolution. Thus, for example, Scorpio currently appears in a particular position in the night sky in summer. In 12,900 years (from now), which is half a period later, Scorpio will be in that same position in the night sky in winter and will not be seen in summer.¹⁰ This effect is called the precession of the equinoxes and was discovered by the Babylonian astronomer Cidenas in 343 BC.¹¹ This apparent motion of the constellations is about one degree per 70 years eastward. More precisely, it constitutes a circular motion about an axis perpendicular to the ecliptic plane passing through the center of the earth. The ecliptic plane contains our sun and its planets, and our moon.

5b. The Earth's Tilted Equatorial Bulge. The precession of the equinoxes is caused by the attractive forces of the sun and the moon on the tilted equatorial bulge of the earth. That bulge is caused by deformation of the elastic earth due to its rotational spin,¹² and the tilt is due to the tilt of the earth's rotational axis (see Fig. 7).

5c. A Tilted Ring Approximates the Bulge. In order to simplify matters we can approximate the earth's shape as that of a sphere plus an equatorial tire-shaped ring of extra mass. The plane defined by this ring

⁷The dot over ϕ is short-hand for "derivative with respect to time": $\dot{\phi} \equiv d\phi/dt$.

⁸See *Classical Mechanics*, H. Goldstein, Addison-Wesley Press, Inc. (1950) and references cited therein.

⁹Note that our calendar year is defined by the seasons (i.e. spring always starting in March).

¹⁰Scorpio will not be seen in the summer night sky 13,000 years from now because it will have moved to the opposite side of the earth which will be in daylight at the time of observation.

¹¹Source: J. B. Marion, *Classical Dynamics of Particles and Systems*, Academic Press 1970.

¹²See "Uniform Circular Motion: Centripetal Force" (MISN-0-17).

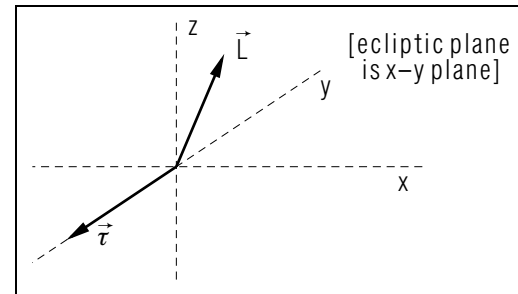


Figure 9. Torque and spin angular momentum vectors for the earth(see text).

will be at an angle of 23.5 to the ecliptic plane. The sun and moon, which appear to move around the earth in circular paths in the ecliptic plane, exert a gravitational torque on the earth-bulge ring, causing the earth's spin vector to precess.

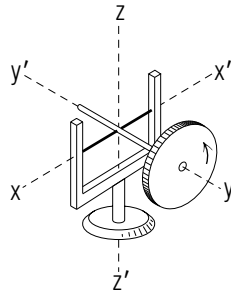
5d. Mass Rings Approximate Sun and Moon Positions. In order to calculate the average torque exerted on the earth-bulge ring by the sun and moon, it is convenient to first set up the average mass distribution of the sun as seen from the earth during the precession cycle. During one such cycle the sun circles the earth 25,800 times and thus in effect can be replaced for this problem by a circular ring of mass, with a radius equal to the earth-sun distance and total mass equal to that of the sun. A similar argument obviously holds for the moon.

5e. Ring-Ring Torque Produces Precession. The approximating sun-moon mass rings exert attractive gravitational forces on the mass of the earth, but in particular produce a torque on the earth-bulge ring as illustrated in Fig. 8. Note that the net gravitational force on any portion of the earth's ring is toward the ecliptic plane, producing a torque in the equatorial plane. This torque and the spin of the earth are shown in Fig. 9, and the resulting precessional motion about the normal to the ecliptic plane can be easily deduced.

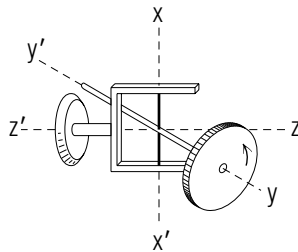
Acknowledgments

We wish to thank Dale Barden for field-testing an earlier version. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

PROBLEM SUPPLEMENT



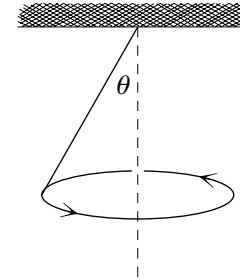
1. a. Describe precession using the motion of a gyroscope rotor, suspended as shown in the sketch, as an example. Be sure to draw in the torque and angular momentum vectors.
- b. Derive the symbolic precessional frequency of the gyroscope.
- c. How does the frequency of precession change as friction slows down the spin rate of a top? Does this agree with your observations of tops and/or gyroscopes?
- d. Assume the moment of inertia of the gyroscope in the problem above is 0.20 kg m^2 and its rotor is spinning at $6.00 \times 10^2 \text{ RPM}$. Calculate the gyroscope's precessional rate and mark its direction of precession on the sketch. The distance from the support to the flywheel is 16 cm and the mass of the flywheel is 8.0 kg .



2. A gyroscope is mounted on a wall, as shown in the sketch. The disk is rotating about the horizontal $y-y'$ axis, as shown. Determine the

axis, direction, and angular velocity of precession. Then determine the frequency of precession. Assume the gyroscope disk has a moment of inertia of $1.00 \times 10^{-3} \text{ lb ft s}^2$, is spinning at $3.00 \times 10^2 \text{ RPM}$, is at a distance of 3.00 inches from the vertical $x-x'$ axis, and has a mass of $0.0300 \text{ lb ft}^{-1} \text{ s}^2$.

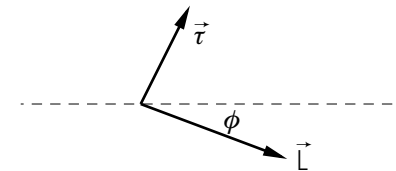
Draw in the torque and spin angular momentum vectors. Sketch the trajectory of the disk.



3. Apply $\vec{\tau} = d\vec{L}/dt$ to the case of a mass hanging from a string and circling the vertical at a constant radius, in order to find the mass's frequency of revolution (here interpreted as precession frequency). Be sure to sketch in the torque and angular momentum vectors.

Brief Answers:

1. a. "Counterclockwise as seen from above..."
- c. "The precession rate increases..."
- d.
$$\begin{aligned} \nu_{\text{prec}} &= \dot{\phi}/(2\pi) \\ &= R_{cg} m g / (4\pi^2 I \nu) \\ &= (0.16 \text{ m}) (8 \text{ kg}) (9.8 \text{ m/s}^2) / (4\pi^2 0.2 \text{ kg m}^2 10/\text{s}) \\ &= 0.16 \text{ rev/s.} \end{aligned}$$



2. Torque is in the z' direction, \vec{L} in the y direction. Then by $\vec{\tau} = d\vec{L}/dt$, precession is about the $x-x'$ axis, initially toward the wall. The trajectory is an arc in the horizontal plane perpendicular to the wall. The rotor hits its support.

$$\dot{\phi} = \frac{\tau}{L} = \frac{FR}{I\omega} = \frac{FR}{I2\pi\nu}$$

Prec. freq. = $\dot{\phi}/2\pi = 1.27 \text{ rev/sec}$

3. $\tau = mgr \sin \theta$

$$L_{\text{horiz}} = L \cos \theta = mrv \cos \theta$$

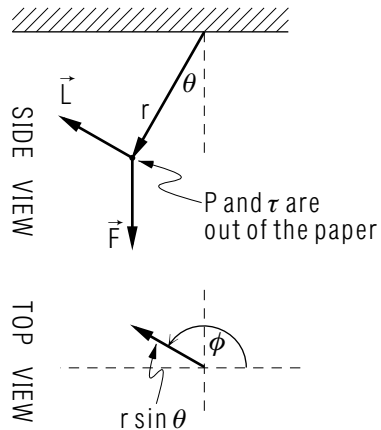
$$\dot{\phi} = \tau/L_{\text{horiz}} = (g \tan \theta)/v$$

Now $v = \dot{\phi}(r \sin \theta)$

hence:

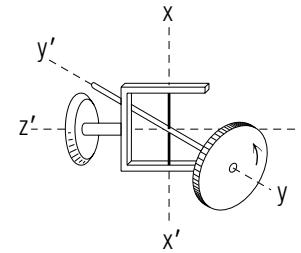
$$\dot{\phi} = \sqrt{g/(r \cos \theta)}$$

(Checks with centrip. force method)¹³



MODEL EXAM

1. See Output Skills K1-K3 in this module's *ID Sheet*. The exam may have one or more of those skills, or none.



2. A gyroscope is mounted on a wall, as shown in the sketch. The disk is rotating about the horizontal $y-y'$ axis, as shown. Determine the axis, direction, and angular velocity of precession. Then determine the frequency of precession. Assume the gyroscope disk has a moment of inertia of $1.00 \times 10^{-3} \text{ lb ft s}^2$, is spinning at $3.00 \times 10^2 \text{ RPM}$, is at a distance of 3.00 inches from the vertical $x-x'$ axis, and has a mass of $0.0300 \text{ lb ft}^{-1} \text{ s}^2$.

Draw in the torque and spin angular momentum vectors. Sketch the trajectory of the disk.

Brief Answers:

1. See this module's *text*.
2. See this module's *Problem Supplement*, Problem 2.

¹³See "Centripetal Force" (MISN-0-17).