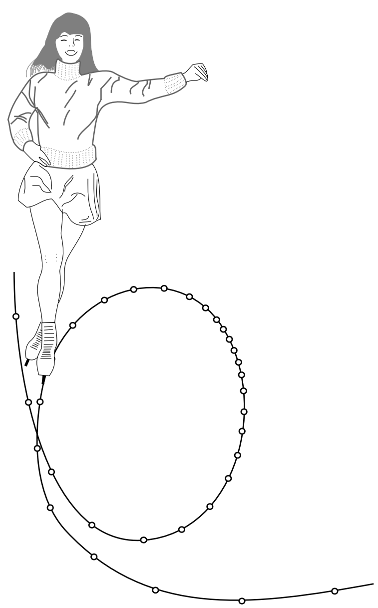
MISN-0-72

TRAJECTORIES AND RADIUS, VELOCITY, ACCELERATION



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

TRAJECTORIES AND RADIUS, VELOCITY, ACCELERATION

by

James M. Tanner, Georgia Institute of Technology

Title: **Trajectories and Radius, Velocity, Acceleration**

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Input Skills:

1. Vocabulary: tangent (to a curve), miles per hour, meter, second, kilometer, parallel, perpendicular, center of curvature, coordinate origin, radians, vector magnitude, vector direction, vector component (MISN-0-2), speed (MISN-0-405).
2. Use symbolic arithmetic; planar vector addition (MISN-0-2).

Output Skills (Knowledge):

K1. Vocabulary: acceleration, acceleration vector, bearing, CCW, CW, kinematics, normal (to), position, position vector, radius vector, speed, time marked trajectory, trajectory, turning rate, velocity, velocity vector, $\vec{r}$, $\vec{v}$, $\vec{a}$.

K2. State the reason why so many topics in science and technology have similar kinematics prerequisites.

Output Skills (Rule Application):

R1. Given a sketch of an object's position, velocity and acceleration vectors at some instant, deduce how the position and velocity vectors are changing. Sketch the trajectory in the neighborhood of this point.

R2. Given an origin, an object's time marked trajectory and a specific point on it; deduce and sketch the position and velocity vectors, including values for length and bearing. Also sketch the acceleration vector's direction, getting it inside the right quadrant, or on the right quadrant boundary, with respect to lines parallel and normal to the velocity vector.

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1. Kinematics and Its Uses

1a. Motion is Everywhere. How many times have you looked around and found everything at rest? A rare experience indeed. And even during such infrequent happenings, there were undetected motions: the beat of your heart; the squirm of microscopic life. Motion, like sound, so surrounds us that its momentary absence seems to suspend the passage of time. In fact one familiar measure of time, the day, depends upon an ever present motion: the earth's rotation about its own axis.

1b. Kinematics Describes Motion. Kinematics is that part of physics which provides concepts and precise mathematical relationships for describing motion. By studying kinematics, you can learn: 1. definitions of kinematical descriptions like velocity, speed and acceleration; 2. procedures for measuring these quantities; and 3. relations between these quantities. As you do this studying, you can practice by analyzing the motions you observe in everyday phenomena.

1c. Dynamics Needs Kinematics. Dynamics, the study of how forces influence motion, relies upon Kinematics to describe that motion.[1] Specifically, Dynamics relates forces to acceleration and then, through kinematics, to changes in an object's position, speed and direction of motion. Thus a good foundation in kinematics is essential to achieving an understanding of how motions and positions are influenced by natural and man-made forces. With such an understanding, motions become not only predictable but designable. The interpretation and design of motion is a recurring theme throughout science and technology, from electrons in integrated circuits to signals in nerves to transfer fluids in energy sources to population flow to climate-changing geophysical changes. This module provides a solid first stage in the acquisition of such capabilities.

[1] See, for example, "Newton's Laws" (MISN-0-14).

2. Time Marked Trajectories

2a. A History of an Object's Motion. As an object moves through space it traces out a path called its "trajectory." The path shown in Fig. 1 is that made by a person ice skating. Dots on the trajectory indicate the skater's position at the end of each second, as might be determined from an overhead multiflash "strobe" photo. Our goal is to get detailed information about motion from such a space-time history of an object's motion.

2b. Deducing Motional Direction. At each instant an object's direction of motion lies along the tangent to its trajectory. For example, in Fig. 2 the tangent to the skater's trajectory at t = 30 seconds has an east-west orientation; the skater's direction of motion is eastward. Similarly, the skater's motion is northward at 10 seconds, westward at 20 seconds, southward at 26 seconds. To determine the precise direction of motion at any time, use a straight edge to draw a tangent line and then use a protractor to measure the angle to a reference direction. You might like to measure the skater's direction of motion at t = 6 sec. and compare

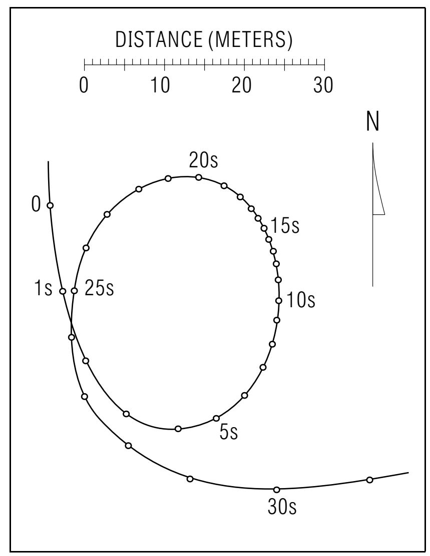


Figure 1. An ice skater's trajectory with positions marked at 1 sec. intervals.

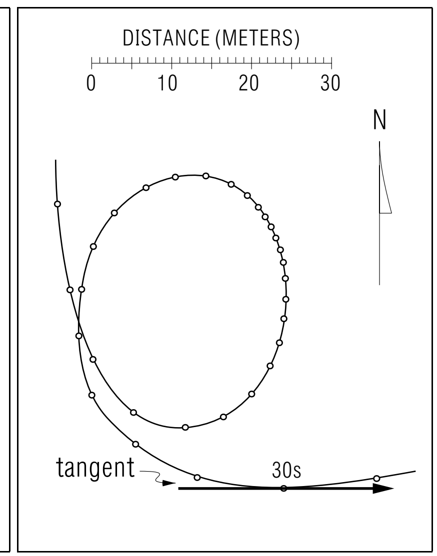


Figure 2. The direction of motion and the tangent to the trajectory of Fig. 1 at the 30 sec. mark.

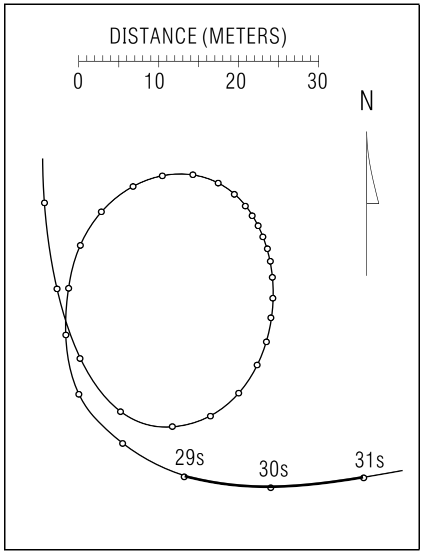


Figure 3. Distance traveled between 29 and 31 seconds, trajectory of Fig. 1.

your answer to ours: N44°E. *Help: [S-1]*[2]

2c. Deducing Speed from the Trajectory. An object's speed, the maginitude of the velocity vector, at any instant can be estimated from its trajectory. For example, in Fig. 3 the skater's speed at 30 seconds can be estimated by measuring the distance traveled between 29 and 31 seconds and then dividing by the elapsed time. To measure this distance, place a thread along the curve and mark the 29 second and 31 second points on the thread. Then straighten the thread along the distance scale. We find a speed of approximately 12 m/s, 27 miles per hour, at the 30 second mark. *Help: [S-2]* Technically, that answer is the average speed over the 2 second interval but we use the general rule that the average of a quantity over an interval is a good estimate of that quantity's value at the midpoint of the interval. Using this method we generated a complete graph of speed vs time (Fig. 4). However the initially decreasing speed, followed by a smooth transition to an increasing speed, is instantly deducible from the variation in the consecutive dot spacings in Fig. 1.[3]

[2]If you need assistance in getting the answer, see sequence [S-1] in this module's Special Assistance Supplement.

[3]Smoothness of the curve results from a requirement that forces have a finite (not infinite) strength.

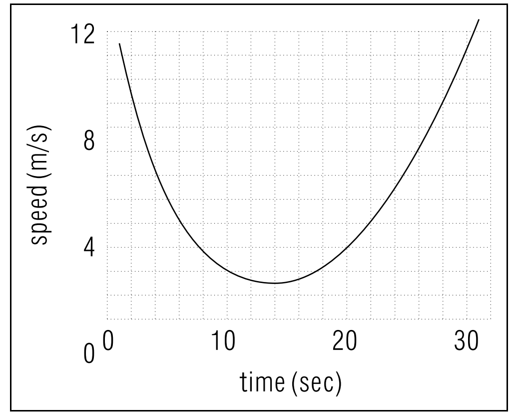


Figure 4. Speed vs time for the trajectory of Fig. 1.

3. The Position Vector $\vec{r}$

If some point is selected as a reference point, an object can be located relative to that point by a vector called the object's "position vector" or "radius vector." For example, in Fig. 5 we have chosen a reference point and labeled it "O."[4] Then the position vector at 15 seconds is the one labeled $\vec{r}$. The position vector's length, the distance from point 0 to the object, is called the vector's "magnitude" and here is 20 meters. The vector's direction is N35°E. As the object moves, its position vector changes in both magnitude and direction. Thus the position vector is a function of time and this is usually indicated by writing it as $\vec{r}(t)$.

4. Velocity: How Position is Changing

4a. Speed, Motional Direction. An object's velocity is a vector which has a magnitude equal to the object's speed and a direction identical to that of the object's motion. Thus, if drawn originating on the object, the velocity vector is tangent to the trajectory as shown in Fig. 6. Velocity is a function of time if speed and/or direction of motion are changing with time. The dependence of velocity on time is symbolized by writing $\vec{v}(t)$. The velocity vector at any one time can be decomposed into components parallel and perpendicular to the position vector in order to determine how the magnitude and direction of the position are changing. This decomposition is illustrated in Fig. 7 for the skater's trajectory at

[4]Reference points are chosen so as to simplify conceptualization or calculation for specific systems. Some examples are: centers of atoms, engine shafts and our planet's sun. Good choices for reference points come with experience and insight.

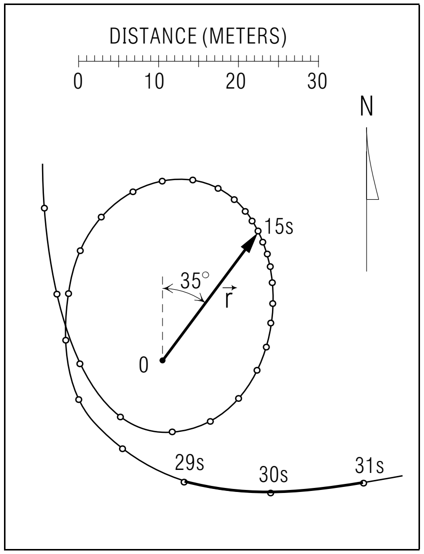


Figure 5. Position vector at 15 sec., trajectory of Fig. 1.

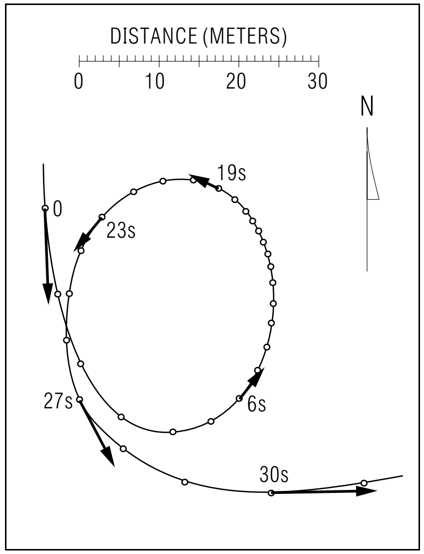


Figure 6. Velocity vectors at various times, trajectory of Fig. 1.

$t = 30$ seconds.

4b. $\vec{v}_{\parallel r}$: $\vec{r}$'s Length is Changing. The direction of the component of velocity parallel to $\vec{r}$, $\vec{v}_{\parallel r}$, indicates whether $\vec{r}$'s length, the distance from the origin, is increasing or decreasing. For an example, see Fig. 7. There, at $t = 30\,\text{s}$, $\vec{v}_{\parallel r}$ is in the same direction as $\vec{r}$, and consequently the magnitude of $\vec{r}$ is increasing. This is verified by observing that at 31 seconds the skater is farther from the origin than at 30 seconds. If $\vec{v}_{\parallel r}$ is opposite to $\vec{r}$ then $\vec{r}$ is decreasing. This is reasonable: motion toward the origin will cause the distance from the origin to shorten. You can illustrate the latter case by constructing $\vec{r}$, $\vec{v}$ and $\vec{v}_{\parallel r}$ at $t = 1$ second on the skater's trajectory. *Help: [S-3]* If $\vec{v}_{\parallel r}$ is zero, the magnitude of $\vec{r}$ is not changing no matter what any other component of $\vec{v}$ has for a value.

▷ Find the four points on the skater's trajectory where $\vec{v}_{\parallel r}$ is zero so the magnitude of $\vec{r}$ is not changing. *Help: [S-4]*

4c. $\vec{v}_{\perp r}$: $\vec{r}$'s Direction is Changing. For two-dimensional motion the component of $\vec{v}$ perpendicular to $\vec{r}$, $\vec{v}_{\perp r}$ is turning clockwise (CW) or counterclockwise (CCW). You can see in Fig. 1 that, for the skater's trajectory, $\vec{r}$ is always turning CCW. In particular, at $t = 30$ seconds (see

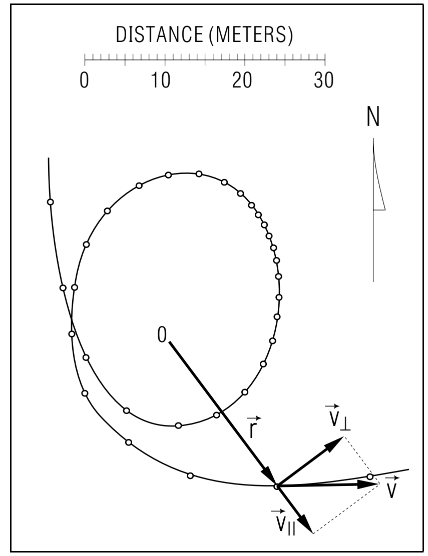


Figure 7. Velocity components at 30 sec., trajectory of Fig. 1.

Fig. 7), $\vec{v}_{\perp r}$ indicates CCW movement; this is verified by the direction of the position vector at $t = 31$ seconds. Any time $\vec{v}_{\perp} r$ is zero, $\vec{r}$ is not changing direction.

▷ Determine whether this occurs at any time on the skater's trajectory in Fig. 7. *Help: [S-5]*

5. Acceleration: How $\vec{V}$ is Changing

5a. Acceleration is to Velocity as Velocity is to Position. Acceleration tells how velocity is changing, entirely analogous to the way velocity tells how position is changing. Thus components of acceleration parallel and normal to the velocity indicate changing speed and changing direction of motion.

5b. $\vec{a}_{\parallel v}$: $\vec{v}$'s Length is Changing. If $\vec{a}_{\parallel v}$ is in the same direction as $\vec{v}$, then velocity is increasing in magnitude. That is, the speed is increasing. Of course, if $\vec{a}_{\parallel v}$ is opposite to $\vec{v}$, then the object's speed is decreasing. If $\vec{a}_{\parallel v}$ is zero, the speed is constant.

5c. $\vec{a}_{\perp v}$: $\vec{v}$'s Direction is Changing. A component of acceleration perpendicular to the velocity indicates that the velocity vector is turning. If the motion is two-dimensional (in a plane), $\vec{v}$ can be specified as turning

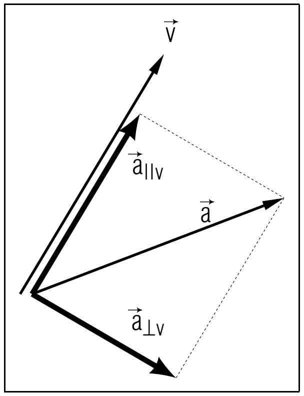


Figure 8. Acceleration components for speed increasing, $\vec{v}$ turning CW.

CW or CCW (See Fig. 8). Thus even though an object's speed may be constant, there can still be an acceleration resulting from a changing direction of motion. Since $\vec{a}_{\perp v}$ is perpendicular to the velocity vector, it always points towards the center of curvature of its neighborhood of the trajectory.

5d. Example: A Skater's Acceleration. Careful inspection of a trajectory permits you to determine the general direction of $\vec{a}$ at any point along the trajectory. For the example in Fig. 1, at $t = 24$ seconds the skater's speed is seen to be increasing and the velocity's direction turning CCW. Therefore, the skater's acceleration has a component in the same direction as the velocity and a CCW-turning component perpendicular to the velocity. These two components and their sum, the acceleration, are shown in Fig. 9.[5]

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

[5]For calculation of the precise vectors shown in Fig. 9, see the Appendix.

Glossary

- $\vec{a}$: symbol for the acceleration vector.
- **acceleration**: the rate of change of a subject's velocity. Accelerations obey the mathematical rules for vectors.
- **acceleration vector**: the time rate of change of an object's velocity vector. The component parallel to the velocity vector, $\vec{a}_{\|v}$, is the rate of change of the velocity vector's magnitude. The component normal to the velocity vector, $\vec{a}_{\perp v}$, is the rate of turning of a unit length vector that continually points along the object's velocity vector.
- **bearing**: direction specified as an angle from either north or south, whichever is closer (e.g. N44°W, S23°E).
- **CCW**: "counter clockwise," the turning direction opposite to CW.
- **CW**: "clockwise," the turning direction of a normal clock's hands.
- **Kinematics**: the discipline concerned with the mathematical description of motion. It is much the smaller of the two subfields of Mechanics. The other, Dynamics, builds upon Kinematics to relate forces to motion. Kinematics has several sometimes-overlapping subdivisions: one dimensional, two dimensional, three dimensional, rotational, non-relativistic, relativistic.
- **normal to**: perpendicular to.
- **position**: the description of an object's location, specified with respect to some coordinate system.
- **position vector**: the radius vector to an object's position.
- $\vec{r}$: symbol for the radius vector.
- **radius vector**: the vector going from a coordinate system origin to a particular point in space.
- **time marked trajectory**: an object's trajectory to which marks have been added showing the position of the object at various times. Often the marks represent equal time intervals, such as would be produced by a stroboscopic lamp flashing at a steady rate.
- **trajectory**: a line showing the path traveled by an object.

- $\vec{v}$: symbol for the velocity vector.
- **velocity**: the rate of change of an object's position. Velocities obey the mathematical rules for vectors.
- **velocity vector**: the time rate of change of an object's position vector. The component parallel to the position vector, $\vec{v}_{\parallel r}$, is the rate of change of the object's distance from the coordinate origin. The component normal to the position vector, $\vec{v}_{\perp r}$, is the turning rate of a unit length vector which continually points along the object's position vector. The length of the velocity vector is called the object's "speed." The direction of the velocity vector is always in the direction of motion.
- **turning rate**: the time rate at which a vector is turning (typically expressed in radians per second).

Deducing $\vec{a}$ (Only for Those Interested)

Given a time marked trajectory, suppose you wish to compute $\vec{a}$ at some point. The module text explains how to determine the directions but not the magnitudes of $\vec{a}$'s components, $\vec{a}_{\parallel v}$ and $\vec{a}_{\perp v}$. Here is a method, illustrated for the 24 second point of Fig. 1:

1. Since $a_{\parallel v}$ is the rate of change of the speed with time, it can be approximated by the change in speed, over some time interval, divided by the length of that time interval. For our example: $a_{\parallel v}(24\,\mathrm{s}) \simeq [v(25\,\mathrm{s}) - v(23\,\mathrm{s})]/(2\,\mathrm{s}) = 0.7\,\mathrm{m/s}^2$.

2. Since $a_{\perp v}$ is the magnitude of the rate of change of the velocity vector due solely to its turning motion, it is given by $a_{\perp v} = v\omega_v$, where ω_v is the velocity vector's turning rate.[6] To estimate this turning rate, measure the angle between the velocity vector directions at two times and divide by the time interval between them. For our example we find an angle of 52.5° between the 23 and 25 second velocity vectors and hence a velocity vector turning rate of: $\omega_v \simeq 52.5°/2\,\mathrm{s} = 26.3°/\,\mathrm{s} = 0.46\,\mathrm{rad/s}$. Radian measure must be used if $a_{\perp v}$ is to have the same units as $a_{\parallel v}$. Then: $a_{\perp v}(24\,\mathrm{s}) = 5.7\,\mathrm{m/s} \times 0.46\,\mathrm{rad/s} = 2.6\,\mathrm{m/s}^2$.

[6] The Greek letter ω, pronounced "oh meg′ ah", is universally used to denote turning rates ("angular velocities"). See "Kinematics: Circular Motion" (MISN-0-9) for typical uses and "Constants of the Motion for Central Forces" (MISN-0-58) for derivations.

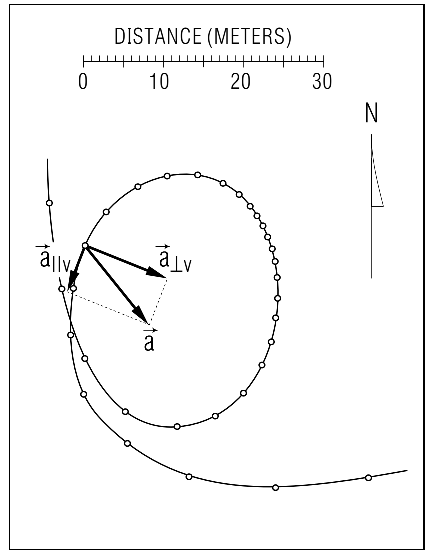


Figure 9. The skater's acceleration at 24 sec. (see Fig. 1).

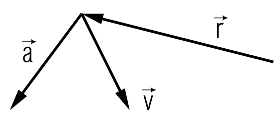
3. Add $\vec{a}_{\parallel v}$ and $\vec{a}_{\perp v}$ to get $\vec{a}$ as in Fig. 9.

PROBLEM SUPPLEMENT

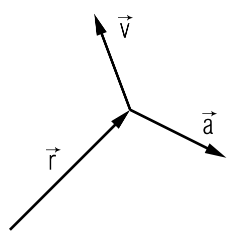
Problems 3 and 4 also occur on this module's *Model Exam.*

1. For the ice skater of Figures 1-5 at 24 seconds:
   a. Determine the radius and positional bearing as seen from the origin in Figure 5.
   b. Determine the speed in m/s and miles per hour.
   c. Determine the bearing of the direction of motion.
   d. On Figure 5, sketch these vectors: $\vec r$, $\vec v$, $\vec v_{\perp r}$, $\vec v_{\parallel r}$, and the directions of $\vec a_{\parallel v}$, $\vec a_{\perp v}$, $\vec a$.
   e. Using only the $\vec r$, $\vec v$ and $\vec a$ vectors of part d, deduce how the radius and velocity vectors are changing.

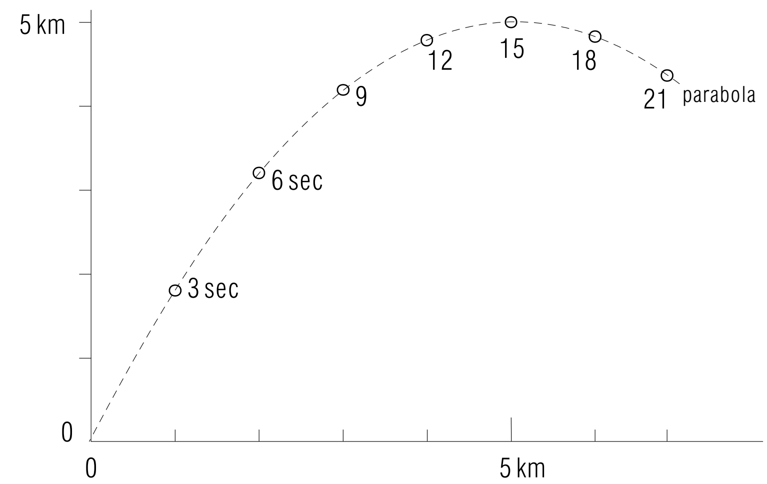
2. Deduce how the position and velocity vectors in the sketch are changing. Show your reasoning. Sketch the trajectory in the neighborhood of this point.



3. Given an object's instantaneous kinematics vectors shown in the sketch, state how the position and velocity vectors' lengths and bearings are changing. Show your reasoning. Sketch the trajectory in the neighborhood of this point.

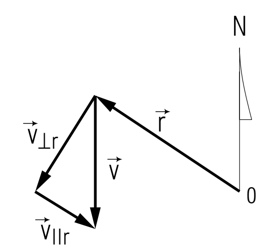


4. Photographs of Jupiter's moon Io show ejection of globs of volcanic material following trajectories similar to that sketched below. For the 18 second point: Determine the lengths and angles of the radius and velocity vectors and sketch them on the figure. Also sketch your best estimate for the direction of the acceleration vector.



Brief Answers:

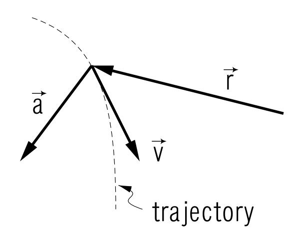
1. For the ice skater:
   a. 18 m, N38°W.
   b. 5.7 m/s = 12.8 mph
   c. approx. S20°W.
   d. Check $\vec a_{\parallel v}$, $\vec a_{\perp v}$, a, on Fig. 9.
   e. The radius vector is moving counterclockwise and is getting smaller, and the velocity vector is increasing both tangentially and radially.



2. $\vec v_{\parallel r}$ is opposite to $\vec r$ so r is decreasing.
   $\vec v_{\perp r}$ shows $\vec r$ is turning CCW *Help: [S-7]*

   $\vec a_{\parallel v}$ is in the same direction as $\vec v$ so v is increasing

   $\vec a_{\perp v}$ shows $\vec v$ is turning CW.

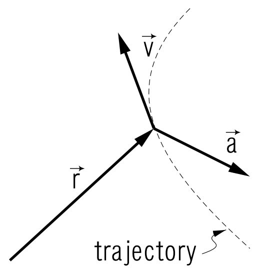


3. $\vec{v}_{\parallel r}$ is in the same direction as $\vec{r}$ so r is increasing.

   $\vec{v}_{\perp r}$ is CCW so $\vec{r}$ is turning CCW.

   $\vec{a}_{\parallel v}$ is in opposite direction to $\vec{v}$ so v is decreasing.

   $\vec{a}_{\perp v}$ is CW so $\vec{v}$ is turning CW.

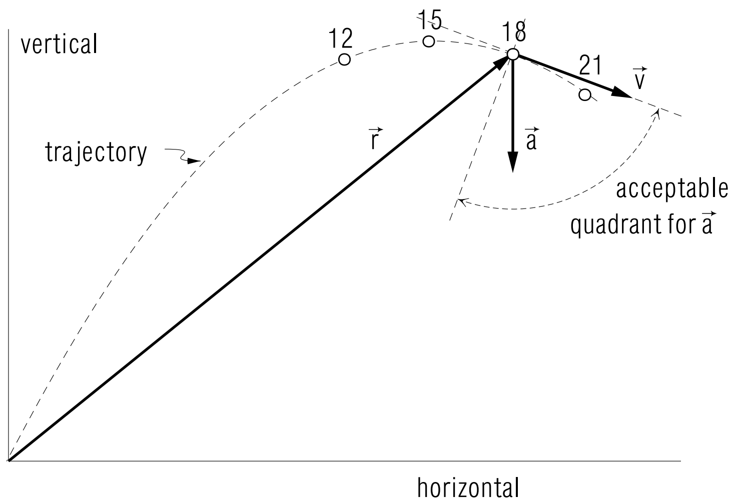


4. $r = 7.6\,\text{km}$

   $\vec{r}$ angle $= 52°$ down from upward vertical

   $v = 0.36\,\text{km/s}$

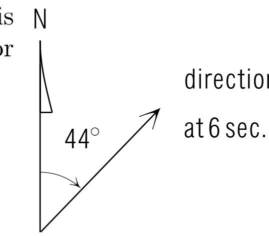
   $\vec{v}$ angle $= 64°$ up from downward vertical.



SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-2b)*

Read N 44° E as "north, 44 degrees east." This means 44° east of northward: *Help: [S-6]* (for assistance with computation of the 44°)

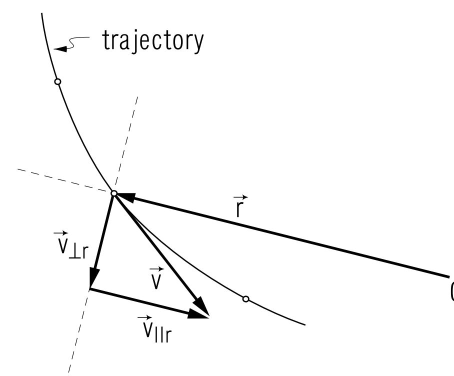


S-2 *(from TX-2c)*

Our thread length was 24 meters on the scale at the top of Figure 3. Then: speed $= 24\,\text{m} / 2\,\text{s} = 12\,\text{m/s}$. You should remember the following conversion which is directly useful on highways in some states and in the rest of the world: $100\,\text{km/hr} = 62.4\,\text{mi/hr}$. Then: speed $= 12\,\text{m/s}(3.6 \times 10^3\,\text{s/hr})(62.4\,\text{mi}/10^5\,\text{m}) = 27\,\text{mi/hr}$.

S-3 *(from TX-4b)*

Note that, for this case, $\vec{v}_{\parallel r}$ is in the opposite direction to $\vec{r}$. Then the magnitude of $\vec{r}$ must be shrinking. Figure 7 shows that this is indeed the case at this time.



S-4 *(from TX-4b)*

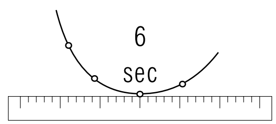
Look for places where the trajectory, hence motion, is normal (perpendicular) to the radius vector: 3 s, 4 s, 20 s, 27 s.

S-5 *(from TX-4c)*

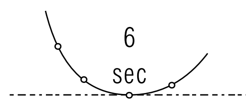
Look for places where the trajectory, hence motion, is straight toward or away from the origin: none.

S-6 *(from S-1)*

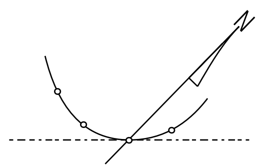
Step 1: Lay a straight edge (a ruler or a folded paper) along the trajectory and rotate it until the angles on both sides of the 6 sec. point look equal.



Step 2: Draw a dashed line along the straight edge, then remove it. Inspect the line.



Step 3: Use the straight edge to draw a northward line through the 6 sec. point.



Step 4: Use a protractor to measure the angle between the two lines or drop a perpendicular to the northward line, measure two legs of the resulting triangle and use their ratio and the appropriate inverse trigonometric function.

S-7 *(from PS-problem 2)*

All radius vectors must radiate out from the coordinate origin. Thus the coordinate origin must be at the tail end of the radius vector shown in the sketch. Note that $\vec{v}_{\perp r}$ points CCW around that coordinate origin. This means that the radius vector is changing in a CCW direction.

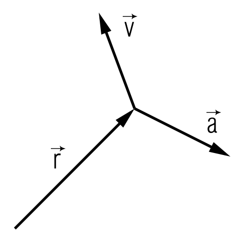
You can see this graphically. Call the radius vector in the sketch $\vec{r}(t)$. Now draw a radius vector from the same coordinate origin out to the trajectory at a slightly later time and label that vector $\vec{r}(t + \Delta t)$. The tails of the two radius vectors are at the same point, the coordinate origin. The tip of the head of $\vec{r}(t + \Delta t)$ is on the trajectory, below and to the right of the tip of $\vec{r}(t)$. It is obvious that rotation from $\vec{r}(t)$ to $\vec{r}(t + \Delta t)$ is in the CCW direction.

MODEL EXAM

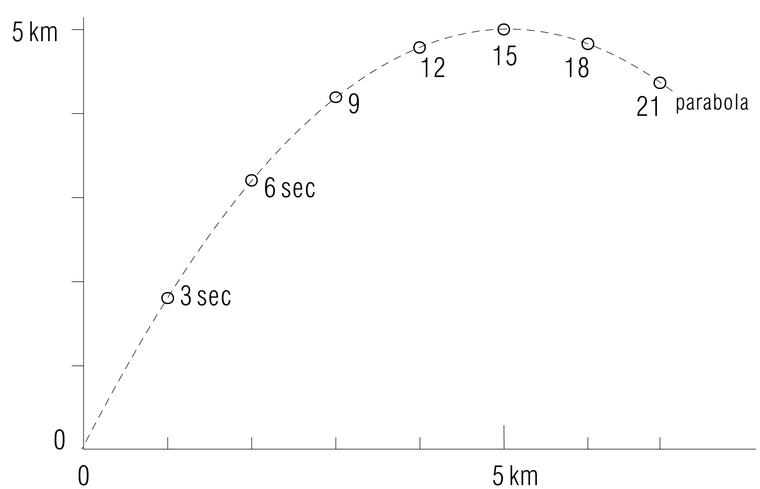
Note: You may bring a protractor, ruler, and string into the Exam Room for exams covering this unit.

1. See Output Skills K1-K2 in this module's *ID Sheet*.

2. Given an object's instantaneous kinematics vectors shown in the sketch, state how the position and velocity vectors' lengths and bearings are changing. Show your reasoning. Sketch the trajectory in the neighborhood of this point.



3. Photographs of Jupiter's moon Io show ejection of globs of volcanic material following trajectories similar to that sketched below. For the 18 second point: Determine the lengths and angles of the radius and velocity vectors and sketch them on the figure. Also sketch your best estimate for the direction of the acceleration vector.



Brief Answers:

1. See this module's *text*.
2. See this module's *Problem Supplement*, problem 3.
3. See this module's *Problem Supplement*, problem 4.