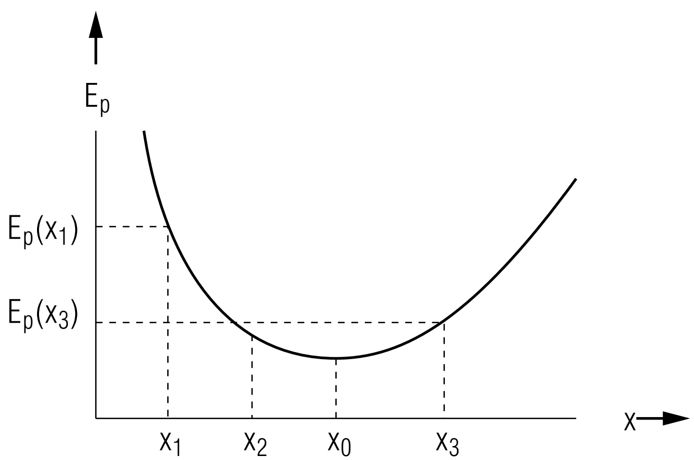
MISN-0-63

POTENTIAL ENERGY CURVES, MOTION, TURNING POINTS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

POTENTIAL ENERGY CURVES, MOTION, TURNING POINTS

by

J. S. Kovacs and P. Signell

Title: **Potential Energy Curves, Motion, Turning Points**

Author: J. S. Kovacs and P. Signell, Michigan State University

Version: 2/1/2000 Evaluation: Stage 0

Length: 1 hr; 16 pages

Input Skills:

1. Given the graph of a function, estimate the slope of the function at any given point.
2. State the law of conservation of energy for mechanical systems, defining kinetic and potential energy (MISN-0-415).

Output Skills (Knowledge):

K1. Vocabulary: potential energy curve, energy diagram, left and right turning points.

Output Skills (Problem Solving):

S1. Given the graph of a one-dimensional potential energy function and the total energy of a particle, give a qualitative description of the motion of this particle and locate its turning points, if any, and regions of acceleration and deceleration.

S2. Given a simple potential energy function for a particle, and the corresponding slope function (in one dimension or radially with spherical symmetry), sketch the function, determine the left and right turning points (if any) of the motion and, for any given position, the force acting on the particle and its acceleration and velocity.

POTENTIAL ENERGY CURVES, MOTION, TURNING POINTS

by

J. S. Kovacs and P. Signell

1. Introduction

1a. Using Potential Energy Instead of Force . In many applications an object's "potential energy" is a more useful dynamical quantity than is the "force" being exerted on the object. In fact, for the quantum world of atoms and molecules the concept of force does not exist and the potential energy function replaces it as the prime quantity of interest. In this module we will work with you on understanding how one uses the potential energy function to deduce motion in classical physics. That means we will illustrate its use with cases where force is also a good concept so you will be able to make a smooth transition in understanding.

1b. What One Can Learn From Potential Energy . If a particle's potential energy graph and its total energy are known, then all dynamical quantities of the particle can be found.[1] For example, from an object's potential energy graph, and the value of its total energy, one can find the force acting on the particle, the particle's acceleration, its velocity and its kinetic energy at any point in space. In this module we show you how to deduce such quantities from a potential energy graph and total energy value. We will restrict ourselves to problems that involve only one coordinate, which means either one-dimensional problems or three-dimensional ones that have spherical symmetry. For the case of sperical symmetry, the single coordinate, r, the distance of the particle from the origin of the force, is sufficient.

2. $F(x)$, $v(x)$, and $a(x)$ from $E_p(x)$

2a. $F(x)$ from Slope of $E_p(x)$. The force on a particle is the negative of the rate at which the particle's potential energy is changing with position. On a graph of potential energy vs position, that is just the negative of the slope of the graph. Thus one can say that the force on a particle, currently passing through some particular point in space, is just the negative of the slope of that graph.

[1] We are here restricting our discussion to conservative, non-dissipative forces, ones that do not include friction.

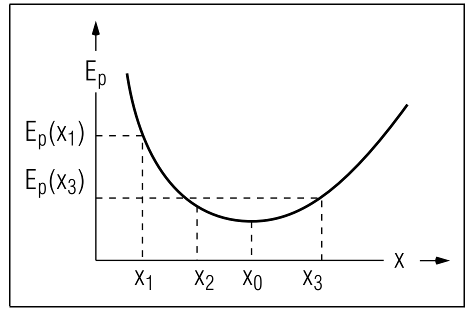


Figure 1. An illustrative potential energy curve (see text).

Characteristics of the force on a particle can be deduced from a graph of the particle's potential energy plotted as a function of position. Consider the potential energy curve in Fig. 1. One immediately notes that the particle with this potential energy is not moving freely. Rather, it feels a force which at each point is the negative of the slope of the $E_p(x)$ curve. At all points for which $x < x_0$ the slope of the curve is negative so the particle feels a positive force, one directed to the right. Furthermore, because the slope at x_1 is steeper than at x_2, the magnitude of the force on the particle is greater at x_1 than at x_2. The farther the particle gets away from x_0, the stronger the force exerted on it. At x_0 the slope is zero so at this point the particle feels no force at all. At all points for which $x > x_0$ the particle feels a negative force, one directed to the left. Note that the force on the particle is always directed back toward x_0, no matter which side of x_0 the particle is on.

2b. $a(x)$ from Slope of $E_p(x)$. Knowing how to deduce the force on a particle from its potential energy graph, one can then use Newton's second law to deduce the particle's acceleration at any particular point in space.

Imagine a particle moving along the x-axis in Fig. 1. Suppose the particle is first observed just to the right of x_0, traveling to the right. At this point the slope is positive and increasing as one moves to the right, so the force is to the left and becoming stronger as the particle moves along. Using $F = ma$, the increasingly negative force implies an increasingly negative acceleration. This increasingly negative (left-pointing) acceleration implies an increasingly strong rate of velocity reduction, an increasingly strong "slowing down" of the particle.

Now suppose the particle reverses its direction and travels left in the region to the right of x_0. The slope is still positive so the force is still negative, meaning the force is still to the left. However, since the particle is traveling to the left, the slope is becoming weaker (see the figure) so the left-pointing force is becoming weaker as x_0 is approached. Then, by $F = ma$, the left-pointing acceleration, which increases the velocity toward the left, is becoming weaker as x_0 is approached. At the instant x_0 is passed, the slope is zero and hence the force and acceleration are zero.

2c. $v(x)$ from Conservation of Energy. The velocity at any point x, $v(x)$, can be obtained from $E_p(x)$ by using the equation for conservation of energy,[2]

$$\frac{1}{2}mv(x_0)^2 + E_p(x_0) = \frac{1}{2}mv(x)^2 + E_p(x). \tag{1}$$

Equation (1) says that the total energy of the particle at point x_0 equals the total energy of the particle at any other point, x. We rearrange Eq. (1) to get:

$$v(x)^2 = v(x_0)^2 + \frac{2}{m}\left(E_p(x_0) - E_p(x)\right) \tag{2}$$

and then take the square root of both sides to get $v(x)$ as a function of $E_p(x)$ and some constants.

3. Deductions From Graphical $E_p(x)$

3a. Energy Diagrams. If both total energy and potential energy are plotted as functions of position on the same graph, useful information about the motion of the particle may be determined from that diagram. The total energy, potential plus kinetic, is always a horizontal straight line on such a graph because the value of the total energy is independent of position. Figure 2 shows the potential energy curve of Fig. 1 with two different total energies superimposed on it. For a given total energy the particle's kinetic energy at each point x can be measured as the vertical distance between the potential energy at that point and the total energy. For example, with total energy E_1, the particle's kinetic energy, when it is at position x_A, is the indicated vertical distance shown on the graph. For a particle with total energy E_2, its potential energy at x_A would be the same as in the E_1 case but its kinetic energy would be less.

[2]See "Potential Energy and Conservation of Energy" (MISN-0-415).

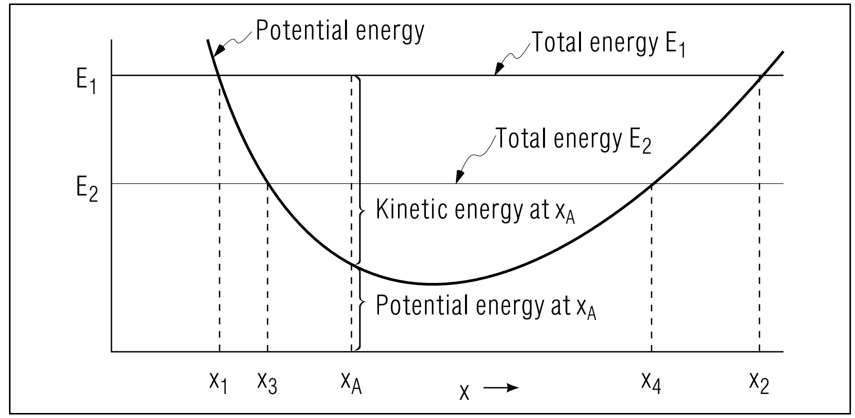


Figure 2. An energy diagram showing one potential energy and two total energy curves on the same graph (see text).

3b. Changes in Speed. Once the kinetic energy E_k is obtained from the energy diagram, the speed can be found:

$$v = \sqrt{2E_k/m}$$

However, the direction of the particle's velocity is ambiguous: the kinetic energy will match the given E_k whether the particle is moving to the right or to the left. However, the particle's change in kinetic energy, and hence its change in speed, does depend on its direction of motion. For example, if the particle in Fig. 2 is at x_A and is moving to the left, its kinetic energy is decreasing so its speed is decreasing. Such changes in speed, which depend on the particle's direction of motion, are given by the direction of its acceleration, the negative of the slope of its potential energy curve.

3c. Turning Points. There are often limits on the positions available to a particular particle: those limits are called "turning points" and they depend on the particle's potential energy function and on its total energy. For example: for a particle with the potential energy function shown in Fig. 2, and with total energy E_1, the particle can go no farther to the left than x_1 and no farther to the right than x_2. Thus x_1 and x_2 are "turning points" for E_1.

Consider the situation in Fig. 2 with the particle at x$_A$, with total energy E_1, and moving left. As it moves left its kinetic energy decreases until the particle reaches x_1 where its potential energy equals its total energy. Since nothing is left for its kinetic energy, its velocity at this point is zero. However, the particle is only instantaneously at rest.

Throughout the neighborhood of x_1 the force acts on the particle to the right, as can be seen from the negative slope of E_p in the neighborhood of x_1. As the particle approaches x_1 from the right, going left, this force causes it to slow down, stop as it reaches x_1, then pick up speed to the right. Thus point x_1 is truly a "turning point" of the motion: it is a point at which the particle stops and "turns around." Similarly, x_2 is a turning point on the right end of the motion: it is the point at which the particle moving to the right comes to a stop, then starts back left. These limits to the motion on the left and right are called the "left" and "right" turning points, respectively.

Note that for total energy E_2 the turning points are x_3 and x_4.

Now suppose the total energy line is lowered until it goes exactly through the minimum of the potential energy curve at point x_0: describe the particle's motion for that case.[3]

4. Dealing with $E_p(r)$

All of the results discussed in this module can be applied to three-dimensional cases that have spherical symmetry. This means cases where the force, potential energy, velocity, etc. depend only on the particle's radius from the origin of coordinates (which is normally the origin of the force associated with E_p). For such cases the one-dimensional position x is replaced by the radial position r. That means that $E_p(r)$ is differentiated with respect to r to get $F(r)$, etc. The main difference from Cartesian coordinates is that negative values of r do not exist.

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

[3] The answer is given in this module's Glossary at the end of the Turning Point discussion.

Glossary

- energy diagram: diagram consisting of the potential energy curve and a horizontal line representing the total energy of the particle.
- potential energy curve, potential energy graph: the graph of the potential energy of a particle as a function of position.
- turning points: positions at which a particle changes its direction of motion. These are positions on an energy vs position graph where the particle's varyiing potential energy line crosses the particle's horizontal total energy line. If, at some point, the potential energy line just touches the total energy line from above, then the left and right turning points coincide and no motion is possble.

PROBLEM SUPPLEMENT

1. A particle traveling in a straight line has a potential energy function $E_p(x) = 7x^2 - x^3$, where the energy is in joules if the particle's position, x, is in meters. The slope of this function is: $14x - 3x^2$. The particle has a mass of 60 kg. Draw a rough sketch of $E_p(x)$. [R] At a certain instant the particle is at $x = +1.0\,\text{m}$ with total energy $E = 36$ joules and is moving to the right.

   a. What is the kinetic energy of the particle at this point? [K]
   b. What is its velocity? [N]
   c. What is the instantaneous force this particle feels at this point? [E]
   d. What is the acceleration (magnitude, direction) this particle undergoes? [I]
   e. Therefore, is this particle's speed increasing or decreasing at this point? [P] Explain. [S]
   f. With this particle moving to the right, and from the shape of the E_p curve in the vicinity of this point ($x = 1.0\,\text{m}$) determine whether the kinetic energy of the particle will increase or decrease. [C] Is this consistent with your answer to part (e)? [L]
   g. For what value of x will the speed of this particle (moving to the right) be zero? (You may not actually be able to find the numerical value of x for which this occurs; however, if you do it right you'll have the equation satisfied by x at this point on the curve you've drawn.) [A]
   h. At this point, where $v = 0$, what is the direction of the acceleration of the particle? (Get this from determining the sign of the force on the particle as determined from the slope of the curve.) [J] With this direction for the acceleration what is the direction of v an instant after the instant when $v = 0$, hence what is the direction of the subsequent motion of the particle? [O]
   i. On your $E_p(x)$ plot, show the region of those values of x forbidden to the particle when it has total energy $E = 36$ joules. [G]
   j. What are the turning points of the motion of the particle when its total energy is 36 joules? [Y]
   k. If at point $x = -3.0\,\text{m}$ a different particle of mass 60 kg starts out with speed $v = 0$:

   (i) What is its kinetic energy? [D]
   (ii) What is its potential energy? [T]
   (iii) What is its total energy? [X]
   (iv) What is the force (magnitude and direction) on the particle? [M]
   (v) What is the magnitude of its acceleration? [V]
   (vi) What is the direction of its acceleration? [B]
   (vii) When it gets to $x = 0$, what is its speed? [H]
   (viii) When it gets to $x = (14/3)\,\text{m}$, what is its speed? [U]
   (ix) What is its acceleration (magnitude and direction) at $x = (14/3)\,\text{m}$? [Q]
   (x) How far will it go before it turns around? [W]
   (xi) Explain your answer to part (x). [F]

Brief Answers:

A. One of the solutions of $x^3 - 7x^2 + 36 = 0$. There are 3 solutions, call them x_1, x_2, x_3 with $x_1 < x_2 < x_3$ (refer to your graph, notice x_1 is negative). x_2 is the answer to this question. The actual values are $x_1 = -2.0\,\text{m}$, $x_2 = 3.0\,\text{m}$, $x_3 = 6.0\,\text{m}$ (you can get this by plotting the graph, by trial and error, and/or by using a hand calculator).

B. To right.

C. Decreases.

D. Zero.

E. $F_x = -11$ newtons (to left).

F. After point $x = (14/3)\,\text{m}$, the potential energy continues to decrease to infinity, total energy remains constant, kinetic energy increases, never goes to zero, so v never changes direction.

G. Forbidden regions: $x < x_1$ and $x_2 < x < x_3$. All other regions are allowed. The above regions are forbidden because the particle would have a negative kinetic energy (not allowed; v^2 cannot be negative) in these regions. Verify this from the energy diagram you've sketched.

H. $\sqrt{3}\,\text{m/s}$.

I. $a_x = -(11/60)\,\mathrm{m/s^2}$ (to left).

J. To left.

K. $E_k = 30$ joules.

L. Yes, kinetic energy and speed both are decreasing.

M. 69 newtons to right.

N. $v_x = +1.0\,\mathrm{m/s}$.

O. Since a_x is to left; an instant after $v_x = 0$, v_x is to the left.

P. Decreasing.

Q. Zero.

R. A rough sketch of $E_p(x)$ can be drawn by considering the following. For very large positive or negative values of x, the x^2 term may be neglected compared to the x^3 term. Hence, for very large negative values the curve tends to plus infinity while for very large positive values the curve goes to negative infinity. The curve crosses the x-axis whenever $E_p(x)$ is zero. There are three roots to a cubic equation. In this case the roots are $x = 0$ (twice) and $x = 7.0\,\mathrm{m}$. You can also determine that the curve has a minimum at $x = 0$ and a maximum at $x = (14/3)\,\mathrm{m}$. With this information a rough graph can be sketched.

S. v_x is positive and a_x is negative, so v_x decreases.

T. 90 joules.

U. $\sqrt{1.14(= 529/405)}\,\mathrm{m/s}$.

V. $1.15\,\mathrm{m/s^2}$.

W. To infinity.

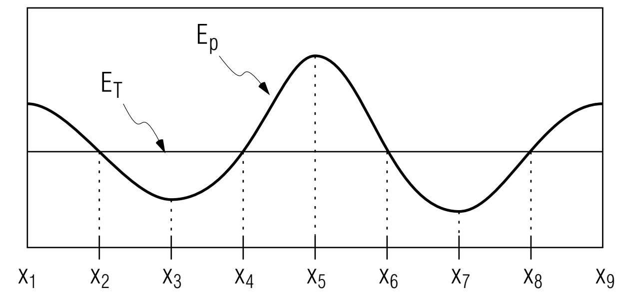
X. 90 joules.

Y. x_1 and x_2: using results of answer (A), the particle moves between points $x = -2.0\,\mathrm{m}$ and $x = +3.0\,\mathrm{m}$.

MODEL EXAM

1. See Output Skill K1 in this module's *ID Sheet*.

2. A particular particle has potential energy E_p and total energy E_T, as shown in the sketch below. It is known that at a particular instant of time the particle is at position x_3. Describe the motion of the particle, including turning points, if any, and regions of acceleration and deceleration.



3. A particular particle of mass 4.0 kg travels in a straight line and has a potential energy of: $2.0 - 4.0x + 4.0x^2$, where the energy is in joules if the particle's position x is in meters. The slope of the potential energy function is: $-4.0 + 8.0x$. The particle has a total energy of: 10.0 J.

   a. Sketch the particle's energy diagram (E_p and E_T vs. x).

   b. Find the particle's x-value at its right turning point.

   c. Find the force on the particle at $x = 2.0\,\mathrm{m}$.

   d. Find the particle's speed at $x = 1.0\,\mathrm{m}$.

   e. Find the particle's acceleration at $x = 0.50\,\mathrm{m}$.

Brief Answers:

1. See this module's *text*.

2. x_2 to x_3: it speeds up
   x_3 to x_4: it slows down
   x_4: it stops momentarily
   x_4 to x_3: it speeds up
   x_3 to x_2: it slows down
   x_2: it stops momentarily

   and the cycle repeats.

3. a. (you can check your sketch by measuring its slope at several points and checking those values against values you calculate from the slope formula given in the problem statement).

   b. 2.0 m.

   c. -1.2×10 N.

   d. 2.0 m/s

   e. 0.