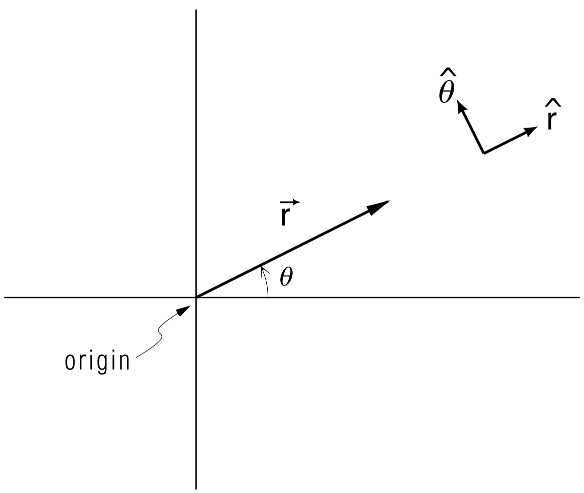
PHYSNET

MISN-0-58

DERIVATION OF THE CONSTANTS OF THE MOTION FOR CENTRAL FORCES



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

DERIVATION OF THE CONSTANTS OF THE MOTION FOR CENTRAL FORCES

by

Peter Signell

Title: **Derivation of the Constants of the Motion for Central Forces**

Author: Peter Signell, Michigan State University

Version: 2/1/2000 Evaluation: Stage 4

Length: 1 hr; 12 pages

Input Skills:

1. State the Law of Conservation of Angular Momentum (MISN-0-41).
2. State the Law of Conservation of Energy (MISN-0-21).

Output Skills (Knowledge):

K2. Separate the polar-coordinate variables in Newton's Second Law and thereby derive the constants of motion, angular momentum and energy, for central forces.

K3. Derive the time derivatives of the polar coordinate unit vectors.

Post-Options:

1. "Derivation of Inverse Square Law Force Field Orbits" (MISN-0-106).

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

DERIVATION OF THE CONSTANTS OF THE MOTION FOR CENTRAL FORCES

by

Peter Signell

1. Introduction

In this module the constancy of angular momentum and energy for central forces is derived from Newton's Second Law in a straightforward manner using vector algebra. Elsewhere the results of this module are used to derive the equations for planetary orbits.[1]

2. Motion In a Plane

We will deal only with motion in a plane, which is the general case for motion of an object in a central force. The coordinates used will be polar, as shown in Fig. 1. Unit vectors will be denoted $\hat{r}$ and $\hat{s}$ as shown in Fig. 1. Thus $\vec{r} = r\hat{r}$. For motion at fixed θ, which is radial motion, both $\hat{r}$ and $\hat{s}$ remain constant (unchanged) as time increases. However if $\hat{s}$ changes with time, then in time dt the unit vector $\hat{r}$ will change by an amount $(d\theta)\hat{s}$ as shown in Fig. 2.

[1]See "Derivation of Inverse Square Law Force Field Orbits" (MISN-0-106).

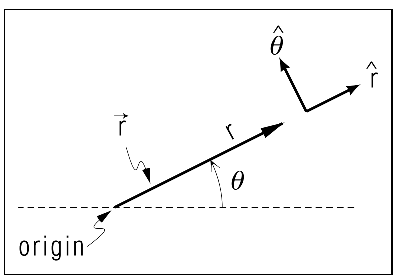


Figure 1. Polar coordinate symbols used here.

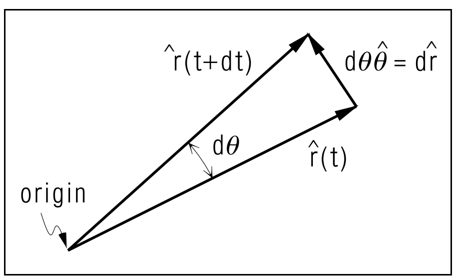


Figure 2. The radial unit vector at two times.

That is: $\hat{r}(t+dt) = r(t) + (d\theta)\hat{s}$. Then:

$$\frac{d\hat{r}}{dt} = \frac{\hat{r}(t+dt) - \hat{r}(t)}{dt} = \frac{d\theta}{dt}\hat{s}.$$

We will henceforth use the "dot" notation for time derivatives. Then the above equation becomes:

$$\dot{\hat{r}} = \dot{\theta}\hat{s}\,.$$

Similarly, you can easily derive that

$$\dot{\hat{s}} = -\dot{\theta}\hat{r}\,.$$

Then returning to $\vec{r} = r\hat{r}$, we can easily find its derivative by taking the derivative of the product:

$$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{s}\,.$$

Taking the derivative again,

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{s} + \dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} - r\dot{\theta}^2\hat{r} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s}\,.$$

3. Central Forces

3a. Definition. A central force is one that is completely radial:

$$\vec{F}_c = f(r,\theta)\hat{r}.$$

Then by Newton's Second Law:

$$\vec{F}_c = f(r,\theta)\hat{r} = m\vec{a} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s}\,. \qquad (1)$$

3b. Angular Component: Conserv. of Angular Momentum. Taking angular components by taking the dot product of both sides of Eq. (1) by $\hat{s}$:

$$0 = m(2\dot{r}\dot{\theta} + r\ddot{\theta})\,.$$

In order to form a complete differential, we multiply both sides by r:

$$0 = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{d}{dt}(mr^2\dot{\theta})\,.$$

Thus the time derivative of $mr^2\dot{\theta}$ is zero, so

$$mr^2\dot{\theta} = \text{ constant;}$$

it is, in fact, the angular momentum about an axis normal (perpendicular) to the plane of motion and passing through the origin:

$$L = mr^2\dot{\theta} = I\omega = \text{ constant.} \qquad (2)$$

3c. Radial Components: Conservation of Energy. Taking radial components by taking the dot product of both sides of Eq. (1) with $\hat{r}$:

$$f(r,\theta) = m(\ddot{r} - r\dot{\theta}^2)\,.$$

We can eliminate $\dot{\theta}$ by solving Eq. (2) for it. Then

$$f(r,\theta) = m\left(\ddot{r} - \frac{L^2}{m^2r^3}\right) = m\ddot{r} - \frac{L^2}{mr^3}\,.$$

If the force function $f(r,\theta)$ is a function of radius alone, then

$$m\ddot{r} - \frac{L^2}{mr^3} - f(r) = 0.$$

In order to form a complete differential we multiply both sides by $\dot{r}$ to get:

$$m\dot{r}\ddot{r} - \frac{L^2}{mr^3}\dot{r} - \dot{r}f(r) = 0. \qquad (3)$$

To see that this is a complete differential, form:

$$v^2 = \vec{v}\cdot\vec{v} = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + \frac{L^2}{m^2r^2}, \qquad (4)$$

so that:

$$\frac{d}{dt}v^2 = 2\dot{r}\ddot{r} - \frac{2L^2}{m^2r^3}\dot{r}.$$

Then Eq. (3) can be written:

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) - \dot{r}f(r) = 0. \qquad (5)$$

The last term can also be converted to differential form by defining:

$$E_p(r) = \int f(r)\,dr,$$

so that:

$$f(r) = -\frac{dE_p(r)}{dr}.$$

Then:

$$\frac{dE_p(r)}{dt} = \frac{dE_p(r)}{dr}\,\dot{r} = -f(r)\dot{r},$$

and Eq. (5) can be written:

$$\frac{d}{dt}\left[\frac{1}{2}mv^2 + E_p(r)\right] = 0.$$

Then:

$$\frac{1}{2}mv^2 + E_p(r) = \text{ constant},$$

where $E_p(r) = -\int f(r)dr$ is the potential energy. The equation is sometimes written in the "radial plus angular kinetic energy" form, obtained by substituting Eq. (4) into Eq. (5):

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + E_p(r) = \text{ constant}.$$

This, in turn, can be written in terms of radial momentum and moment of inertia as:

$$\frac{p_r^2}{2m} + \frac{L^2}{2I} + E_p(r) = \text{ constant}. \qquad (6)$$

Then Eq. (6) is the sum of radial kinetic, angular kinetic, and potential energy.

4. Conclusion

The appearance of Conservation of Angular Momentum from the angular equation and Conservation of Energy from the radial equation is very interesting. A whole different formulation, called the Hamiltonian formalism, uses such ideas as its basis. Identification of the constants of motion is thereby greatly simplified, especially in cases of complex motions.

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

MODEL EXAM

1. See Output Skills K1-K3 in this module's *ID Sheet*. Credit may be received by taking a written exam (closed book, no notes) or by means of a lecture (closed book, one 3"×5" card of notes). For lecture credit, there must be no gaps, erasures, or pauses. Be sure to include diagrams in either exam mode.