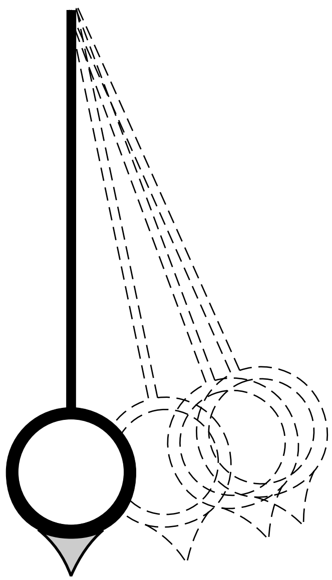
MISN-0-42

PENDULA, SIMPLE AND PHYSICAL



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

PENDULA, SIMPLE AND PHYSICAL

by

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Title: **Pendula, Simple and Physical**

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Version: 2/1/2000 Evaluation: Stage 0

Length: 1 hr; 16 pages

Input Skills:

1. Write the power series expansions about zero for the sine and cosine functions (MISN-0-4).
2. Identify the differential equation associated with simple harmonic motion and extract the period and frequency of the motion from this equation (MISN-0-26).
3. Apply the equations of rotational motion to determine the angular acceleration of an extended object constrained to rotate about one of its principal axes (MISN-0-36).

Output Skills (Knowledge):

K1. Starting from the free-body diagram showing all of the forces on the object, derive the expression which is the differential equation of periodic motion of a rigid body (point mass on massless rod or extended object) suspended from a fixed point in the gravity field of the earth. In the small-displacement approximation, show that this is the equation for simple harmonic motion and find then the period associated with this motion.

Output Skills (Problem Solving):

S1. Given a particular simple or physical pendulem, calculate the period for small oscillations.

Post-Options:

1. "Translational plus Rotational Motion of a Rigid Body" (MISN-0-43).
2. "Ideal Collisions Between a Frictional Sphere and a Flat Surface: The Superball" (MISN-0-53).
3. "Euler's Equations; The Tennis Racket Theorem" (MISN-0-57).

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New authors, reviewers and field testers are welcome.

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1. Introduction

1a. Periodic Oscillation of Suspended Objects. An object suspended in the gravitational field of the earth hangs there in some equilibrium position (such that the center-of-mass of the object is directly below the fixed point of suspension). If this object is displaced from this equilibrium position and released, its motion is simple harmonic motion - where the period associated with the vibratory motion is independent of the amount of displacement form equilibrium and depends only upon the distribution of the mass of the object and the constant acceleration due to gravity. The general expression for the period of this vibration will be derived and applied to some simple systems.

1b. The Simple Pendulum. A simple pendulum, a mathematical idealization, is defined as a point mass at the end of a rigid massless rod suspended in the gravitational field of the earth free to oscillate about the vertically oriented equilibrium position. The motion about the equilibrium position is simple harmonic for small displacements from the equilibrium with period

$$P = 2\pi\sqrt{\frac{L}{g}}. \tag{1}$$

Here L is the distance from the point of pivot to the mass.

1c. The Physical Pendulum. Any object suspended from a fixed point in the object, free to pivot about some horizontal axis through that point, will execute oscillatory motion about its equilibrium orientation. The equilibrium orientation is that for which the static equilibrium conditions are satisfied.[1] That occurs when the center of mass of the suspended object is directly below the point of pivot.

[1]The conditions of static equilibrium are satisfied when the resultant force on the object is zero and the resultant torque of these forces relative to any point in space is zero. See "Static Equilibrium, Centers of Force, Gravity, and Mass" (MISN-0-6).

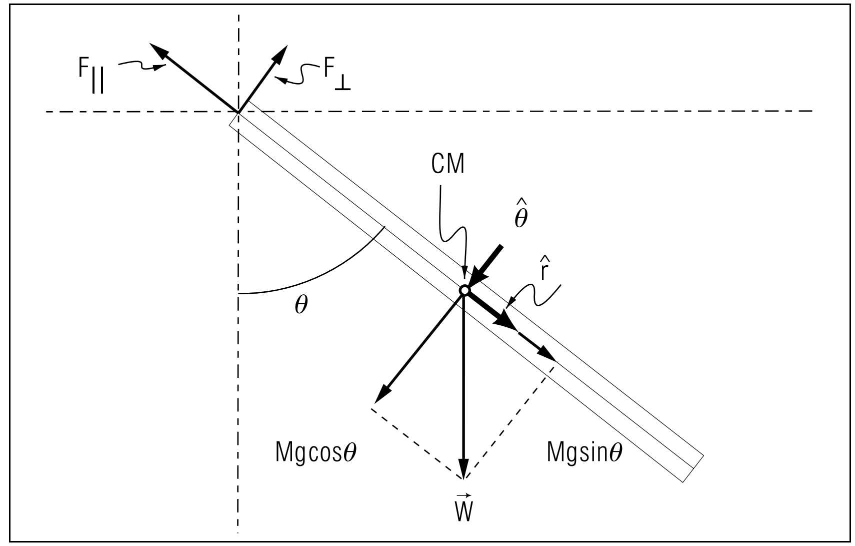


Figure 1. A rod as a physical pendulum. The point of application of $\vec{F}$ denotes the PIN point. The unit vector $\hat{z}$ points upward, out of the page.

2. Analysis of the Physical Pendulum

2a. Example: Force and Torque on a Suspended Uniform Rod. As a concrete example, consider a uniform rod of length L, mass M, fixed at one end so that it is free to rotate in a vertical plane. At some instant in its oscillatory motion what are the forces acting on this object? There are essentially two: the force of gravity, $\vec{W}$, acting at the center of mass, and the force, $\vec{F}_{PIN}$, the axis exerts on the rod at the pivot point (See Fig. 1). To facilitate expressing the forces and torques acting on this system (the rod), let us set up a coordinate system with a set of orthogonal unit vectors associated with this coordinate system. Instead of the usual orthogonal coordinate system fixed in space, it will be more convenient to use a coordinate system that is attached to the rod.[2]

Consider a set of unit vectors attached to the CM of the rod: $\vec{r}$ is parallel to the rod, pointing in the direction in which $\vec{r}$ increases ($\vec{r}$ being the vector from the origin of the coordinates, the point where the rod

[2]Such a coordinate system has to be used with caution, because relative to an observer who watches the rod move, these unit vectors are not fixed in direction.

is attached, to any point on the rod, or in space for that matter), $\hat{\theta}$ is perpendicular to the rod (tangent to the circle swept out by the center of mass) and pointing in a direction in which θ increases. The third unit vector, $\hat{z}$, points up, out of the page and is defined by the cross-product of the other unit vectors: $\hat{z} = \hat{r} \times \hat{\theta}$. In Fig. 1 the rod's motion is restricted to the plane of the paper. The unit vectors $\hat{r}$ and $\hat{\theta}$ are always in this plane so $\hat{z}$ is always perpendicular to it. Relative to this framework, we can express the forces on the rod as:

$$\vec{F}_{PIN} = -F_{\parallel}\hat{r} + F_{\perp}\hat{\theta}, \tag{2}$$

$$\vec{W} = (Mg\cos\theta)\hat{r} - (Mg\sin\theta)\hat{\theta}. \tag{3}$$

Relative to the point where the rod is pinned, what are the torques on the system? Obviously, the torque of $\vec{F}_{PIN}$ is zero. The torque of the gravity force can be shown to be:

$$\vec{\tau} = -(Mg\,b\sin\theta)\hat{z}, \tag{4}$$

where b is the distance from the origin to the center of mass of the object ($b = L/2$ in this case). The torque $\vec{\tau}$ is directed into the page, tending to rotate the system clockwise about the axis through the pin.

2b. The Equation of Motion for the Uniform Rod. The motion is about a principal axis (an axis perpendicular to the rod at the upper end), therefore $\vec{L}$ and $\vec{\omega}$ are parallel: $\vec{L} = I\vec{\omega}$. It is always true that:

$$\frac{d\vec{L}}{dt} = \vec{\tau},$$

so that, with $\vec{L} = I\vec{\omega}$ we have:

$$\vec{\tau} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}.$$

This latter relation[3] is correct only if the direction of $\vec{\tau}$ is parallel to a principal axis (as it is in this case). What is the direction of the vector $\vec{\alpha}$? Because the motion of the object is planar, $\vec{\alpha}$ can only be perpendicular to that plane (recall that the tangential acceleration of a point undergoing circular motion is given by $\vec{a}_T = \vec{\alpha} \times \vec{r}$). Therefore, in the case of this system:

$$\vec{\alpha} = \frac{d^2\theta}{dt^2}\,\hat{z} \tag{5}$$

[3]Recall Sect. 1 of "Rotational Motion of a Rigid Body" (MISN-0-36).

(If $d^2\theta/dt^2$ is positive, $\vec{\alpha}$ is up, out of the page.) The second derivative, $d^2\theta/dt^2$, is what we are seeking. If we know it as a function of time and integrate it twice, then we'll have a complete description of the angular motion of the object (if we know two initial conditions as well). The torque equation above, which we derived starting from Newton's Law, tells us how this effect, the acceleration $d^2\theta/dt^2$, is related to the external torque through $\vec{\tau} = I\vec{\alpha}$. For our rod, then:

$$-(Mg\,b\sin\theta)\,\hat{z} = \left(I\frac{d^2\theta}{dt^2}\right)\,\hat{z} \tag{6}$$

$$\left(\frac{d^2\theta}{dt^2} + \frac{Mg\,b\sin\theta}{I}\right)\,\hat{z} = 0, \tag{7}$$

a vector that equals zero. It will be zero, if its components (in this case only the one component) have zero magnitude:

$$\frac{d^2\theta}{dt^2} + \frac{Mg\,b\sin\theta}{I} = 0. \tag{8}$$

The quantity I is the moment of inertia of the object relative to the axis of rotation.

2c. The Small Angular Displacement Approximation. The solution to this second order differential equation gives θ, the angular displacement from the equilibrium position ($\theta = 0$), as a function of time. This seemingly simple differential equation does not have a solution that is easily expressible in terms of known functions. However, if the restriction is made that the displacement from equilibrium for all values of t is "small," then this equation falls into a familiar form. What is "small"? By this is meant the condition under which you can to an accurate approximation replace $\sin\theta$ by θ. From the power series definition of $\sin\theta$,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$$

we see that this condition is satisfied if θ^3 is insignificant when compared with θ (in radians). Reference to a table of trigonometric functions verifies that, to three significant figures, the values for $\sin\theta$ and θ do not begin to differ appreciably until the angle gets greater than 10°. With this approximation, valid for small θ, the differential equation becomes:

$$\frac{d^2\theta}{dt^2} + \frac{Mg\,b}{I}\theta = 0. \tag{9}$$

2d. Solution to the Small Angle Equation of Motion. Equation (9) has well known and simple solutions, linear combinations of sines and cosines, for example:

$$\theta(t) = A\sin(\omega t) + B\cos(\omega t) \tag{10}$$

where: $\omega = (M g\, b/I)^{1/2}$ and A and B are constants depending on initial values (at t=0) of θ and $d\theta/dt$.[4] That this is a solution may be verified by substituting it into the differential equation.

2e. General Expression for the Period of Oscillation. The periodicity associated with this motion can be ascertained by looking at the phase of the sine or cosine function. The phase should be $(2\pi/p)t$ or $(2\pi\nu)t$, where p and ν are the period and frequency associated with the motion. Hence the period of this pendulum is:

$$P = 2\pi\sqrt{\frac{I}{M g\, b}}. \tag{11}$$

Note that the period of this oscillatory motion depends upon the value of the acceleration of gravity at the location of the pendulum. A pendulum clock, calibrated at one location, will be in error if moved to another location where g is different.

2f. Periods of the Uniform Rod and Simple Pendula. Specifically, for the uniform rod of length L, free to rotate about an axis perpendicular to the rod at one end, the moment of inertia, using Steiner's (the Parallel Axis) Theorem, is given starting from:

$$I = I_c + Ma^2, \tag{12}$$

where I_c is the moment of inertia with respect to an axis through the center of mass parallel to the pivot axis ($ML^2/12$ for a uniform rod) and a is the distance from the CM to the point about which the desired moment of inertial is sought. Hence,

$$I = \frac{ML^2}{3}.$$

With $b = L/2$, the period of oscillatory motion is

$$P = 2\pi\sqrt{\frac{2L}{3g}}$$

[4]See MISN-0-26 for an example.

For the simple pendulum, $I = ML^2$, $b = L$, so that

$$P = 2\pi\sqrt{\frac{L}{g}}$$

as stated in Eq. (1).

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

PROBLEM SUPPLEMENT

Note: Problem 1 also occurs in this module's *Model Exam*.

1. A pendulum consists of two point particles of mass $0.40\,M$ and $0.60\,M$, respectively, attached to a rigid massless rod of length L. The rod is pinned at one end, and is thereby constrained to move in a vertical plane in the gravity field of the earth. One mass, $0.40\,M$, is located one-fourth of the rod's length from the pinned end. The other mass, $0.6\,M$ is at the free end.

   a. Locate the center of mass of the system. [C]

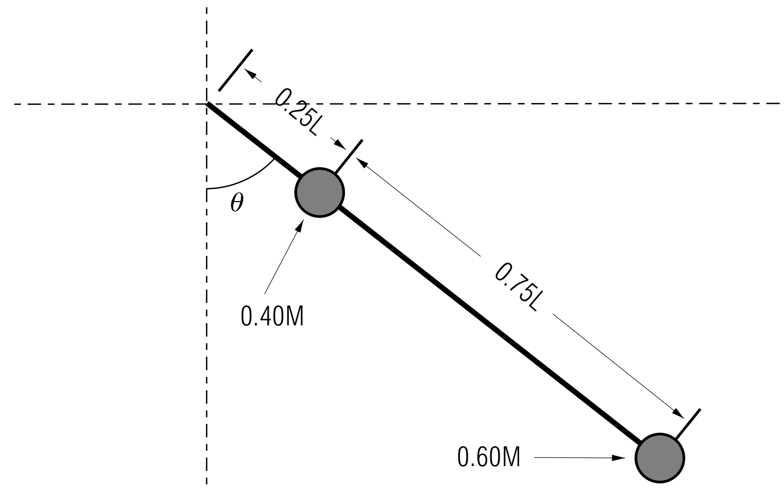
   b. Draw a free-body diagram showing all of the forces acting on the system (the rod). [B]

   c. Relative to the fixed end of the rod find the net torque on the system, and from this find the differential equation which describes the angular motion of the rod. [D]

   d. Find the moment of inertia of this system. [F]

   e. What is the period of oscillatory motion (for small oscillations) of this system? [A]

   f. If the fixed pivot point were midway between the two masses, what should be the period of small oscillations? [E] What should happen to the period as the fixed point approaches the CM? [G]



Brief Answers:

A. $2\sqrt{50L/56g}$.

B. See textual material.

C. (7/10) L from fixed end.

D. See textual material.

E. $2\pi\sqrt{30L/16g}$.

F. $(5/8)M\,L^2$.

G. It should get larger, eventually approaching infinity at the CM.

MODEL EXAM

1. Without referring to a text or notes, derive the expression for the period of the simple pendulum for small oscillations. If you wish to follow a guide which emphasizes the important features that should be covered in the derivation, carry out the development in these steps:

    a. On a sketch of the pendulum showing the system inclined at an angle θ with the vertical, show all of the forces acting on the point mass.

    b. Use the definition of torque to determine the torque on the mass relative to the point suspension. Express your result in terms of the set of unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{z}$, where $\hat{r}$ is the unit vector along the direction from the origin to the point mass and $\hat{\theta}$ is perpendicular to this in the direction of increasing θ.

    c. Determine the angular acceleration of the pendulum at this angle θ.

    d. From the resulting equation, making a suitable approximation, identify the frequency of oscillatory motion. Explain under what conditions it is correct.

2. A pendulum consists of two point particles of mass $0.40\,M$ and $0.60\,M$, respectively, attached to a rigid massless rod of length L. The rod is pinned at one end and is thereby constrained to move in a vertical plane in the gravity field of the earth. One mass, $0.40\,M$, is located one-fourth of the rod's length from the pinned end. The other mass, $0.6\,M$ is at the fre end.

    a. Locate the center of mass of the system.

    b. Draw a free-body diagram showing all of the forces acting on the system (the rod).

    c. Relative to the fixed end of the rod find the net torque on the system, and from this find the differential equation which describes the angular motion of the rod.

    d. Find the moment of inertia of this system.

    e. What is the period of oscillatory motion (for small oscillations) of this system?

f. If the fixed pivot point were midway between the two masses, what should be the period of small oscillations? What should happen to the period as the fixed point approaches the CM?

Brief Answers:

1. See this module's *text*.
2. See this module's *Problem Supplement*, problem 1.