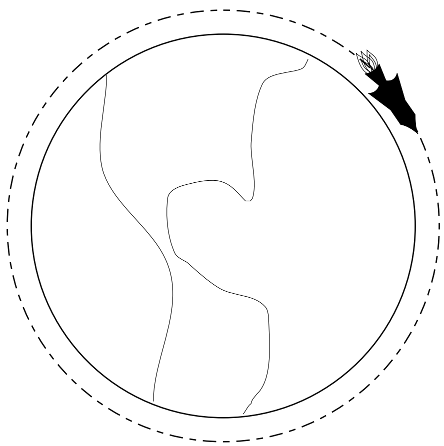
PHYSNET

MISN-0-41

DYNAMICS FOR CIRCULAR MOTION



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

DYNAMICS FOR CIRCULAR MOTION

by

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Title: **Dynamics for Circular Motion**

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Input Skills:

1. Vocabulary: circular motion, frequency, angular velocity (MISN-0-9); kinetic energy, power (mechanical), work (mechanical) (MISN-0-20).
2. State the relation between angular velocity and tangential velocity in circular motion (MISN-0-9).
3. State the work-energy principle and the force-power relation for constant velocity (MISN-0-20).

Output Skills (Rule Application):

R1. Given a planar configuration of rigidly connected masses and an axis perpendicular to its plane, calculate its moment of inertia about the axis.

Output Skills (Problem Solving):

S1. Use, and justify the use of conservation of angular momentum to solve problems involving torqueless change from one state of uniform circular motion to another.

S2. Solve problems involving energy of rotation in circular motion, starting from the equation for kinetic energy in linear motion.

S3. Solve problems involving power transfer in uniform circular motion, starting from the equation for power transfer in uniform linear motion.

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OF PROJECT PHYSNET

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Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

Views expressed in a module are those of the module author(s) and are not necessarily those of other project participants.

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1. Introduction

1a. The Law is Useful. The easiest way to achieve a quick understanding of much natural and man-made rotational phenomena, and predictive power as well, is through the Law of Conservation of Angular Momentum. As an example of application to a rotating system, consider a figure skater spinning with angular velocity ω, arms outstretched. The skater suddenly draws her arms inward and straightens her body and thereby produces a dramatic change in her angular velocity. Using the Law of Conservation of Angular Momentum, we can not only predict whether her new spin rate will be less or greater than her old one, but also make a good quick estimate or a precise calculation of her actual change in angular velocity. The same technique used in this case can be applied to a huge number of diverse systems.

1b. The Law Appears to be Exact and Universal. The Law of Conservation of Angular Momentum is one of the great laws of the universe. It has been observed to hold, to the limits of measurable accuracy, for the incredibly small particles which live only virtually and for incredibly brief times within the hearts of atoms, to hold for everyday natural and man-made objects, and to hold for the stars and galaxies in distant parts of the universe. It is on the same sort of footing as the Law of Conservation of Momentum, in that no violation of it has ever been found.

2. Magnitude of Angular Momentum

2a. Definition: Point Mass, Circular Motion. The symbol L is universally used to represent the magnitude of mechanical angular momentum: its definition for a point mass m in circular motion is:

$$L = mvr \qquad \text{(point mass in circular motion).} \qquad (1)$$

The parameters m, v, and r are illustrated in Fig. 1. The rotating mass m could be a tiny chunk at a radius r in a compact disc, or it could represent the mass of one of a spinning ice skater's arms at an effective radius r. It could be a molecule at a radius r from the center of a centrifuge, an

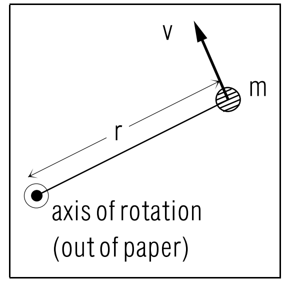


Figure 1. Parameters used in discussing angular momentum.

electron at a radius r from the center of its atom, or it could be a car on a turn, with radius of curvature r, in the Indy 500. For each of these, we take the product of the object's mass, speed, and radial distance from the axis about which it is rotating.

In reality, angular momentum is a vector quantity, hence it is often written $\vec{L}$. However, only its magnitude L, defined above, is needed for dealing with planar motion.[1]

2b. The Additive Property of Angular Momentum. The angular momentum of a collection of masses is just the sum of the angular momenta of the individual masses. This is especially simple to state mathematically for cases where the masses all move in the same plane. For two such masses:

$$L = m_1 v_1 r_1 + m_2 v_2 r_2 \qquad \text{(planar motion).}$$

Here both radii must be measured from the same axis of rotation. In general, for N masses, the additive property is written:

$$L = \sum_{i=1}^{N} m_i v_i r_i \qquad \text{(planar motion).}$$

2c. Rigid Objects: Moment of Inertia. The angular momentum of a rigid planar object is often written as the product of the object's angular velocity and its "moment of inertia." Objects for which this is appropriate include the record, the skater, the centrifuge, and the car cited above. Such a rigid object's component masses all have the same

[1] For the full vector definition see the Appendix.

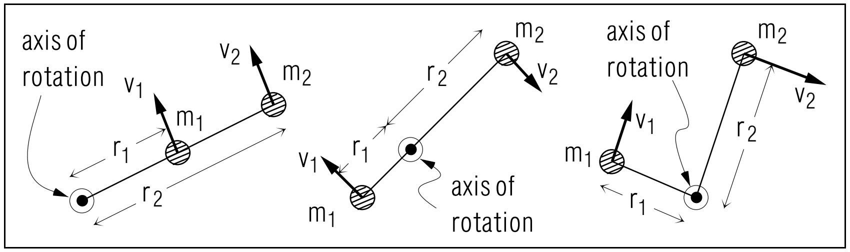


Figure 2. Examples of rigid planar rotators having two masses. The masses' radii are measured from a common axis of rotation.

angular velocity and thus the angular momentum can be written:

$$L = \sum_{i=1}^{N} m_i v_i r_i = \sum_{i=1}^{N} m_i \omega r_i^2 = (m_1 r_1^2 + m_2 r_2^2 + \ldots)\,\omega = \left[\sum_{i=1}^{N} m_i r_i^2\right] \omega .$$

The factored quantity in the brackets is determined solely by the spatial distributions[2] and values of the component masses and is independent of the state of rotation of the object. It is always denoted by the symbol I:

$$I = \sum_{i=1}^{N} m_i r_i^2 \qquad \text{(planar set of masses)}.$$

the quantity I is called the object's "moment of inertia." The angular momentum is then written:

$$L = I\omega \qquad \text{(rigid planar object)},$$

which is the rotational analogue of the expression for linear momentum,

$$p = mv \qquad \text{(linear motion)},$$

with $p \to L$, $m \to I$, $v \to \omega$. The same substitutions occur in other equations that are rotational analogues of linear equations.

3. Conservation of Angular Momentum

3a. Torque. An object in circular motion is said to be experiencing a torque with respect to its axis of rotation if it is experiencing a force with

[2]Spatial distribution $\equiv$ positions in space.

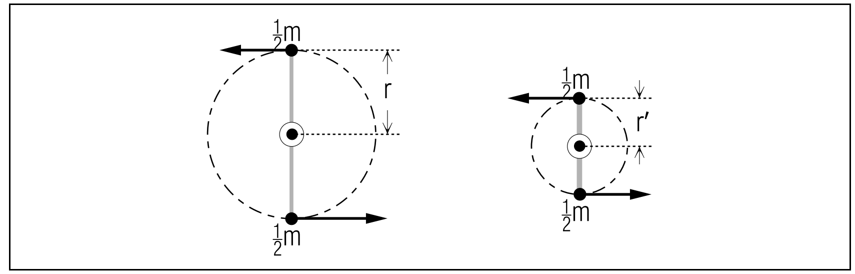


Figure 3. Looking down on the figure skater before and after the arms are pulled in. Here r and r' are the body mass's effective radii, before and after.

a component along the direction of motion. This checks with our common meaning of the "torque." More precisely, the magnitude of torque is given by:

$$\tau = rF_\perp ,$$

where $F_\perp$ is the component of force perpendicular to the object's radius vector. Torque is actually a vector quantity although only its magnitude is needed for dealing with planar motion.

3b. Statement of the Conservation Law. Here is the law of Conservation of Angular Momentum:

> As long as an object experiences no torque about an axis, the object's angular momentum about that axis will be conserved.

3c. An Example: The Spinning Skater. Consider the spinning ice skater who brings in her arms and straightens up her body in order to increase her spin rate: her angular momentum about her vertical spin axis will be conserved if she experiences no torque about that axis. Now her only point of contact with an external object which could produce such a torque is at the skate-to-ice surface. However, this is a very small point and is well lubricated with pressure-melted ice water: it can exert only an exceedingly small amount of torque. Thus we can say that her angular momentum is conserved to a high accuracy as she moves her body to increase her spin rate. Using primes for quantities measured after the body movements, as in Figure 3, the conservation law is:

$$L = L' ,$$

hence:

$$\omega' = (r/r')^2\omega\,.$$

If the skater cuts her mass's effective radius by a factor of two, she could increase her spin rate by a factor of four! You can demonstrate this effect quite dramatically using a rotating lab stool with heavy weights in your hands that heighten the change in your mass's effective radius.

4. Rotational Kinetic Energy & Power

4a. Energy of Rotation: The Flywheel. Suppose we have a flywheel, perhaps one in an experimental car,[3] and we wish to know how much energy we can store in it for a given angular velocity. If the flywheel has almost all its mass m on its rim, then that mass is essentially all at a single radius r and its rotational kinetic energy is:

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2\,.$$

This is the energy one must put into the flywheel to take it from rest to angular velocity ω. It is also the energy one can get back by stopping the flywheel, so one can say that it is the energy stored in the flywheel by virtue of its rotation. Note that the equation for rotational kinetic energy can be generated from the one for linear kinetic energy by making these replacements: $m \to I$, $v \to \omega$.

4b. Power Supplied by a Constant Torque In UCM. A real flywheel has frictional energy losses, and perhaps others as well, and thus it requires a driving force in order to maintain a constant angular velocity. The power that needs to be transferred into the system by a constant driving force F acting at a point moving with constant velocity v is:[4]

$$P = \frac{d}{dt}E = \frac{d}{dt}(Fx) = Fv. \qquad (\text{constant } F)\,.$$

For our case the force is to act at a radius r and is to be always perpendicular to the rotational axis (for example, the force exerted on a flywheel by a belt.) The quantity $r \times F$ is the magnitude of the torque, τ, acting on the flywheel. Since r and F are constant in our example, the torque exerted on the flywheel will be constant. This is UCM so our flywheel is rotating at a constant angular velocity ω. Putting these all together,

$$P = Fv = F\,r\,\omega = \tau\,\omega. \qquad (\text{constant } \tau)\,.$$

This energy flows in as mechanical work and out as frictional heat. Note that the conversion from linear to angular quantities can be carried out, as usual, through the replacements: $F \to \tau$, $v \to \omega$.

[3] See "An Overview of Dynamics"(MISN-0-62) and "New Flywheel Designs for Energy Storage" (MISN-0-46, Learner Originates).

[4] See "Work and Power" (MISN-0-20).

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

A. Vector Angular Momentum

Only For Those Interested. When dealing with non-planar motion, one must represent angular momentum by a vector, not just the magnitude of a vector. This makes Conservation of Angular Momentum into a much more powerful tool for the scientist or engineer.

For an object containing masses $m^{(1)}$, $m^{(2)}$ ... at positions $\vec{r}^{\,(1)}$, $\vec{r}^{\,(2)}$, ..., and having velocities $\vec{v}^{\,(1)}$, $\vec{v}^{\,(1)}$, ... :

$$\vec{L} = \sum_i \vec{r}^{\,(i)} \times \vec{p}^{\,(i)}.$$

where the vector cross product is indicated and the masses' momenta are given as usual by $\vec{p}^{\,(i)} = m^{(i)}\,\vec{v}^{\,(i)}$. The other relationships in this module, in their full vector glory, are (with n and m representing Cartesian components and x_n representing components of $\vec{r}$):

$$I_{nm} = \sum_i m^{(i)} x_n^{(i)} x_m^{(i)}\,; \qquad L_n = \sum_m I_{nm}\omega_m\,; \qquad \vec{\tau} = \vec{r} \times \vec{F}\,,$$

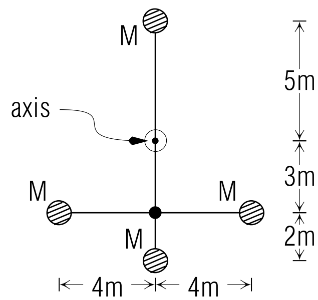
$$\frac{d\vec{L}}{dt} = \vec{\tau} \qquad (\text{consv. of Ang. Mom. when } \tau = 0)\,,$$

$$P = \vec{F} \cdot \vec{v}. \qquad (\text{constant } \vec{F})$$

PROBLEM SUPPLEMENT

Problems 5-8 also occur on this module's *Model Exam.*

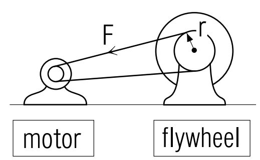
1. Calculate the moment of inertia, of the set of rigidly connected masses shown, about the axis of rotation shown. Each mass M is 2.0 kg and the distances shown are: 2.0 m, 3.0 m, 4.0 m, and 5.0 m.



2. An ice skater is spinning at 3.00 rev/sec about a vertical axis with his arms close to his body. He then changes his moment of inertia from $1.20\,\mathrm{kg\,m^2}$ to $1.60\,\mathrm{kg\,m^2}$ by extending his arms straight outward. Use conservation of angular momentum to determine his new spin rate.

3. Starting from the equation for rotational kinetic energy, calculate the change in the skater's kinetic energy in Problem 2 above. State whether the change was a loss or a gain for the skater. Where do you think the increment of energy came from or went to?

4. A motor must deliver 1.5×10^2 W to maintain a flywheel at a constant 6.00×10^2 rpm. If the drive belt's maximum surface strength against abrasion is 5.0×10^1 N, calculate the minimum radius for a "drive pulley" that can be used for the belt.



   Start from the equation for power delivered by a constant force to maintain constant linear velocity.

5. If four masses ($m = 2.0$ kg each) are at the corners of a 6.0 m × 8.0 m rectangle, find their collective moment of inertia about an axis perpendicular to the plane of the rectangle and passing through its center. Sketch the geometry.

6. A particular phonograph turntable and disc have a combined moment of inertia of $(1/64)\,\mathrm{lb\,ft\,sec^2}$ and are rotating freely at 30.0 rpm (rev/min). A fly with mass $(1/32)\,\mathrm{lb\,sec^2/ft}$, flying in the same direction as a fresh coffee spot on the disc, but at half the speed of the spot, lands on the spot which is 6.0 inches from the center of the disc.

   a. Justify and use conservation of angular momentum to calculate the new rotational frequency of the disc.

   b. Are the numbers in this problem realistic?

7. Two sky divers are positioned so their bodies are parallel and they are face-to-face in free fall toward the earth. They have equal moments of inertia about the single axis passing through both of their centers of mass, and one of them is rotating about this axis at twice the rotational frequency of the other. They now join hands and rotate as a unit. Justify and use conservation of angular momentum to calculate the percentage loss or gain in mechanical energy due to the hand joining. *Help: [S-1]* Where does this energy come from or go to?

8. A particular jet has four engines, each of which develops 1.8×10^4 N of thrust (force) at 7.46×10^6 W. Calculate the minimum period in days for the jet to make one revolution around the earth. (The radius of the earth is 6.37×10^6 m.)

Brief Answers:

1. $2.0 \times 10^2\,\mathrm{kg\,m^2}$

2. 2.25 rev/sec

3. loss of 53 J of energy: goes to heat in arm muscles. *Help: [S-2]*

4. 0.048 m *Help: [S-2]*

5. $2.0 \times 10^2\,\mathrm{kg\,m^2}$

6. a. Since the turntable turns freely, no external torque can be exerted on it; so angular momentum is conserved.

$$\nu' = \frac{1 + (x/2)}{1 + x}\nu\,,$$

   where $x \equiv mr^2/I = 1/2$; then $\nu' = 25$ rpm. *Help: [S-4]*

   b. A fly that weighs one pound?

7. 1.0×10^{1}% (i.e., 10%) lost to heat in arm muscles, eventually transferred to the surrounding air. *Help: [S-1]*

8. 1.1 day *Help: [S-3]*

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-problem 7)*

Read the first sentence in the problem word by word and make sure you have the orientation of the two sky divers *exactly as described there* rather than according to some preconceived idea of sky diver orientation. If you still can't do this problem, work problems 1 and 2 again.

S-2 *(from PS-problems 3 and 4)*

What are the units of angular velocity?
What are the units of frequency?
Which one is given in the problem?
Which one occurs in the equation you are using?
What is the conversion factor between them?
If you are unable to answer any of the above questions, see the Input Skills reference in this module's *ID Sheet* or see the Volume's *Index*.

S-3 *(from PS-problem 8)*

Think to yourself: What equations do I know that relate time (the desired quantity) to one or more of the given quantities (power, force, distance)? Then, if that equation or set of equations has unknowns, what equations do I know that relate those unknowns to given quantities?
Also:

$$\frac{(4)(1.8\times10^{4}\,\mathrm{N})(2\pi\times6.37\times10^{6}\,\mathrm{m})}{(4)(7.46\times10^{6}\,\mathrm{W})(\,\mathrm{N\,m/W})(3600\,\mathrm{s/hr})(24\,\mathrm{hr/day})}=1.1\,\mathrm{day}$$

S-4 *(from PS-problem 6a)*

We suggest that you just go ahead and add the angular momenta of the fly and the turntable before the fly alights. Leave any unknown quantity as a symbol. If necessary, relate velocity to frequency and radius. Then do the same for the angular momenta after the fly alights, realizing that the fly and the turntable now have the same rotational frequency. Equate the "before" angular momentum to the "after" angular momentum and solve for the "after" frequency.

MODEL EXAM

1. If four masses ($m = 2.0$ kg each) are at the corners of a 6.0 m $\times$ 8.0 m rectangle, find their collective moment of inertia about an axis perpendicular to the plane of the rectangle and passing through its center. Sketch the geometry.

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   a. Justify and use conservation of angular momentum to calculate the new rotational frequency of the disc.

   b. Are the numbers in this problem realistic?

3. Two sky divers are face to face in free fall toward the earth. They have equal moments of inertia about the single axis passing through both of their centers of mass, and one of them is rotating about this axis at twice the rotational frequency of the other. They now join hands and rotate as a unit. Justify and use conservation of angular momentum to calculate the percentage loss or gain in mechanical energy due to the hand joining. Where does this energy come from or go to?

4. A particular jet has four engines, each of which develops 1.8×10^4 N of thrust (force) at 7.46×10^6 W. Calculate the minimum period in days for the jet to make one revolution around the earth. (The radius of the earth is 6.37×10^6 m.)

Brief Answers:

1-4. See this module's *Problem Supplement*, problems 5-8.