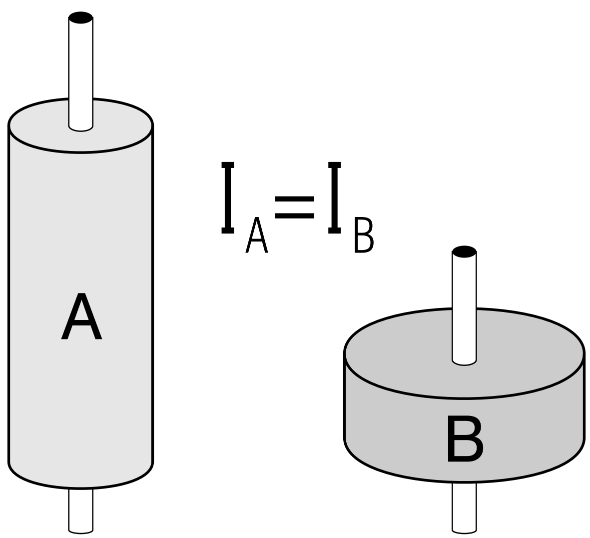
PHYSNET

MISN-0-35

# MOMENTS OF INERTIA, PRINCIPAL MOMENTS




Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

MOMENTS OF INERTIA, PRINCIPAL MOMENTS

by

J. S. Kovacs

Title: **MOMENTS OF INERTIA, PRINCIPAL MOMENTS**

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**Input Skills**:

1. Use the scalar product to determine the angle between two given vectors (MISN-0-2).
2. Set up and solve integrals in one, two, and three dimensions (MISN-0-1).
3. State the law of conservation of angular momentum (MISN-0-41).

**Output Skills (Knowledge)**:

K1. Define moment of inertia, principal axes of inertia, and principal moments of inertia.

K2. Define the radius of gyration of a rigid body.

**Output Skills (Project)**:

P1. Given the angular velocity and the angular momentum vectors of a rotating rigid body with respect to some fixed reference system, determine whether or not the axis of rotation is one of the principal axes of the body. Also determine the moment of inertia of this object with respect to this rotation axis.

P2. Calculate the moment of inertia relative to a given axis of simple mass configurations when the mass and the distribution of mass are given. Also, given the radius of gyration, determine the moment of inertia of a rigid body.

P3. Use the conservation of angular momentum principle to determine the moment of inertia of an irregularly shaped object when a change in location of a part of the object of known moment of inertia produces an observed change in angular velocity.

**External Resources (Required)**:

1. M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading (1970). For availability, see this module's *Local Guide*.

# MOMENTS OF INERTIA, PRINCIPAL MOMENTS

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## 1. Introduction

In general, the motion of a rigid body consists of a translation (straight line or some other trajectory motion) of the center of mass plus a rotation of the object about some axis through the center of mass. In this unit we treat the second part alone, the case of rotation only. Therein, we develop the relationship between the angular momentum and angular velocity vectors. If, for some rotation, these two vectors are parallel, the axis of rotation is said to be a "principal axis." Involved in these relationships is the concept of *moments of inertia*. We consider how to calculate the moment of inertia of a given object from the definition of that quantity.

## 2. $L$ and $\omega$ for a Rotating Rigid Object; Principal Axes of Inertia

**2a. Discussion.** A point mass moving along a circular path has an angular velocity vector, $\vec{\omega}$, directed along the axis of the circle, and an angular momentum vector, $\vec{L}$ , relative to the center of the circle which is parallel to the angular velocity. The quantities are related in magnitude by $L = MR^2\omega$ where $M$ is the particle mass and $R$ is the radius of the circle. The combination $MR^2$ is the moment of inertia of the point mass relative to the axis of rotation.[1] An extended rigid body may be viewed as a distribution of point masses. If such a rigid body rotates about some fixed axis, the angular velocity vector and the angular momentum vector are not, in general, parallel. However, a relation between $\vec{\omega}$ and the component of $\vec{L}$ which is parallel to $\vec{\omega}$ can still be written down. The proportionality factor is *the moment of inertia of the rigid body relative to the axis of rotation*,

$$L_\omega = I_\omega \omega, \tag{1}$$

[1]These quantities are defined and related in "Torque and Angular Acceleration for Rigid Planar Objects: Flywheels" (MISN-0-33). See also "Kinematics: Circular Motion" (MISN-0-9).





where $L_\omega$ is the component of $\vec{L}$ in the same direction as $\vec{\omega}$. That is,

$$\vec{L}\cdot\vec{\omega} = L\omega\cos\alpha \equiv L_\omega \omega,$$

$\alpha$ being the angle between $\vec{L}$ and $\vec{\omega}$. $I$ is the *moment of inertia of the rigid body relative to the axis of rotation* determined by the vector $\vec{\omega}$. For an extended rigid object it is the analog of what $MR^2$ is for a point object of mass $M$ moving in a circle of radius $R$. In fact, by considering the rigid body as a collection of point masses, each its own distance from the axis of rotation, one can directly calculate the moment of inertia for the object under consideration.[2] The other components of $\vec{L}$ (those perpendicular to $\vec{\omega}$) cannot be related to $\vec{\omega}$ via a relation such as Eq. (1).

Suppose that $\vec{L}$ is *parallel* to $\vec{\omega}$. Then $\vec{L}$ has no components perpendicular to $\vec{\omega}$, and Eq. (1) can be written as a *vector equation*:

$$\vec{L} = I\vec{\omega}. \tag{2}$$

The conditions under which Eq. (2) is satisfied usually exist if the angular velocity $\vec{\omega}$ is directed along one of the symmetry axes of the object.[3] These are called the *principal axes of inertia* of the object and the moments of inertia relative to these axes are the *principal moments of inertia*.

As an illustration of these concepts, consider the following situation. Suppose that relative to a fixed inertial reference frame the angular momentum vector of a rotating rigid body is given by:

$$\vec{L} = [A\sin(\alpha)\sin(\omega t+\delta)]\hat{x} + [A\sin(\alpha)\cos(\omega t+\delta)]\hat{y} + [A\cos\alpha]\hat{z}.$$

Here $A$, $\alpha$ and $\delta$ are constants, $t$ is the time, and $\omega$ is the magnitude of the angular velocity of the rotating rigid body. The vector $\vec{\omega}$ has magnitude and direction given by $\vec{\omega} = \omega\hat{z}$. The quantities $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors along the coordinate axes of the inertial reference frame. It is clear that $\vec{L}$ and $\vec{\omega}$ are not parallel ($\vec{\omega}$ does not have $x$- and $y$-components, $\vec{L}$ does). The axis of rotation of this object is thus not one of the principal axes of inertia.

It can be verified that the magnitude of $\vec{L}$ is $A$, the angle between $\vec{L}$ and $\vec{\omega}$ is $\alpha$, and that the moment of inertia of this object relative to the $z$-direction is $(A/\omega)\cos\alpha$. No other moment of inertia for this body can be determined from the information supplied.

[2]This calculation is the subject of the next section of this module.

[3]However, even if an object has no symmetry associated with it, it can be proven that there are at least three mutually perpendicular directions for which Eq. (2) is satisfied if the rotation is about one of these axes.

**2b. Additional Study Material.** In AF[4] study Sections 11.1 and 11.2 (p. 210-213), especially Example 11.1 and Fig. 11.6.

## 3. Moment of Inertia by Two Methods

Learn to get the moment of inertia: (1) by direct calculation; and (2) by conservation of angular momentum.

**Study**: In AF, Section 11.3 (pp. 213-217), Section 10.5 (pp. 188-189). For access, see this module's *Local Guide*.

## 4. The Radius of Gyration

If all of the mass of an object were located at a point instead of being distributed as it is, where should all of this mass be so that the moment of inertia of the mass relative to some axis is the same as for the object itself? The distance from the axis to this point is called the *radius of gyration*, and is usually designated by $k$. The moment of inertia of the object is related to the radius of gyration by:

$$I = Mk^2,$$

as defined for a point mass $M$.

As an example, the radius of gyration of a sphere of radius $R$ relative to an axis through its center is:

$$k = \sqrt{0.4}R \simeq 0.632R.$$

## Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

[4] M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading (1970). For access, see this module's *Local Guide*.





## LOCAL GUIDE

The readings for this unit are on reserve for you in the Physics-Astronomy Library, Room 230 in the Physics-Astronomy Building. Ask for them as "The readings for CBI Unit 35." Do **not** ask for them by book title.

## PROBLEM SUPPLEMENT

1. For Fig. 11.6 in AF[5] verify:

   a. The magnitude of $\vec{L}$ is $A$.

   b. The angle between $\vec{L}$ and $\vec{\omega}$ is $\alpha$.

   c. The moment of inertia of the object with respect to the $z$-axis is numerically equal to $(A/\omega)\sin\alpha$.

2. For a rotating rigid body the angular velocity it has relative to a fixed coordinate system is given by $\vec{\omega} = \omega_x\hat{x} + \omega_y\hat{y}$, where $\omega_x$ and $\omega_y$ are constants. Its angular momentum relative to this same set of axes is given by: $\vec{L} = L_x\hat{x} + L_y\hat{y}$. With $L_x$ written as $A\omega_x$ and $L_y$ written as $B\omega_y$, determine the following:

   a. Is the axis of rotation of this object "fixed"?

   b. Are $\vec{L}$ and $\vec{\omega}$ parallel?

   c. What is the moment of inertia of this rigid body with respect to the given axis of rotation?

   d. Is the axis about which the rotation is taking place a principal axis of rotation for the object?

   e. Under what condition (expressed in terms of the coefficients $A$ and $B$) will this axis be a principal axis for the object?

   f. What is then the principal moment of inertia relative to this axis?

3. How would you determine the moment of inertia of an irregularly shaped object such as, say, your own body? You can't calculate it by direct integration as you would, for example, for a cylinder, relative to its axis. You can, however, use the principle of angular momentum conservation to determine it experimentally.

   Consider that you wish to determine your moment of inertia relative to a vertical axis through the center of mass of your body (when you are in a standing position). If you stand on a rotating turntable rotating at a constant angular speed about a vertical axis, the component of your angular moment along the axis of rotation is proportional to the moment of inertia you want to determine.

[5] M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading (1970). For access, see this module's *Local Guide*.





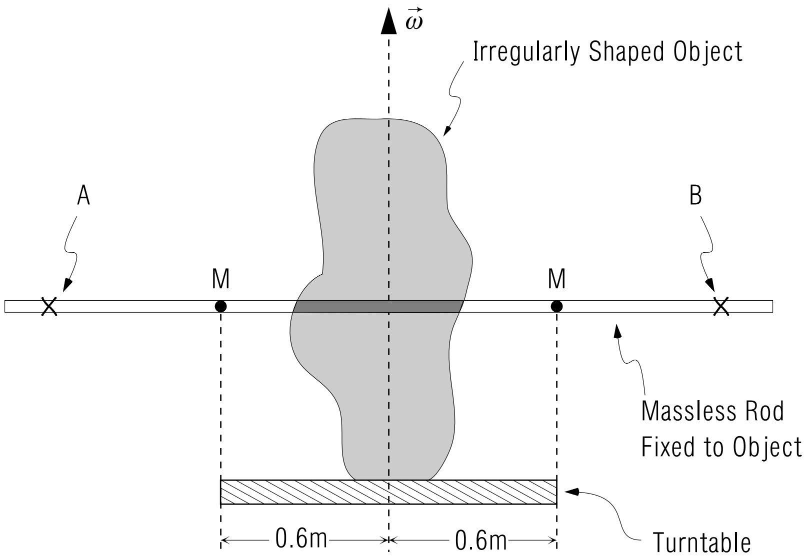



**Figure 1.** Irregularly shaped object rotating about vertical axis on a turntable which rotates with very little frictional loss. Two masses $M$ are located on a massless rod held horizontally fixed to the object.

The problem is to extract it from the angular momentum which is also unknown. This can be done by causing a known change in the moment of inertia of the system (you and anything you might be holding) and observing the change in angular velocity. For example, Suppose you are holding a long rod of negligible mass horizontally across your body. Attached to this rod and free to slide along the rod are two 4 kilogram masses (small enough to be treated as point-like), symmetrically placed on the rod at opposite sides of your body each 0.6 meters from the vertical axis through your body (See Figure 1). The masses are temporarily held fixed at the 0.6 meter location. The system (you and the loaded rod) is observed to rotate at 30 revolutions per minute. At some instant the masses are released and move out to point $A$ and $B$ in Fig. 1, each point being 1.2 m from the vertical axis. With this rearrangement the system is observed to now rotate at 10 revolutions per minute. Determine your moment of inertia from this data.

4. In AF work problems 11.1, 11.2, 11.3 and 11.4.
5. a. For the person on the turntable described in problem 3, determine the radius of gyration with respect to the vertical axis if the person weighs 160 pounds (determine the radius of gyration in meters).

   b. Determine the radius of gyration of the person plus masses-on-the-rod system when the masses are at points $A$ and $B$.

**Brief Answers**:

2. a. Yes, $\vec{\omega}$ is constant.

   b. Not in general. Depends on values of $A$ and $B$.

   c. $\dfrac{A\omega_x^2 + B\omega_y^2}{\omega_x^2 + \omega_y^2}$.

   d. No, $\vec{L}$ and $\vec{\omega}$ aren't parallel for arbitrary $A$ and $B$.

   e. If $A = B$ then $\vec{L} = A\vec{\omega}$ and they are parallel.

   f. $A$.

3. $1.44\,\mathrm{kg\,m}^2$.
4. AF 11.1 Radius of gyration, when axis is at one end, is $0.612\,\mathrm{m}$.

   AF 11.2 a. $I = 1.94\,\mathrm{kg\,m}^2$, $k = 0.611m$.

   AF 11.2 b. $I_{\mathrm{rod}} = 48ML^2$ where $M = 0.20\,\mathrm{kg}$ is the total mass of the rod and $L = 1.0\,\mathrm{m}$ is the length of the rod. For the system, rod plus masses, $I_{\mathrm{total}} = O.966\,\mathrm{kg\,m}^2$, $k = 0.431\,\mathrm{m}$.

   AF 11.2 c. $I_{\mathrm{total}} = 0.642\,\mathrm{kg\,m}^2$, $k = 0.351\,\mathrm{m}$.

   AF 11.3 Answers in AF are correct. For part (c), recall that the center of mass of the equilateral triangle is at the intersection of the medians (see Table 4.1, p. 47) which is two-thirds of the distance from either vertex to the midpoint of the opposite side.

5. Answers: (a) $0.14\,\mathrm{m}$; (b) $0.4\,\mathrm{m}$.





# MODEL EXAM

Note: This Model Exam consists of more detailed step-by-step questions than the actual exam is likely to have. Consider it both as a Model Exam and a Review.

1. See Output Skills K1-K2 on this module's *ID Sheet*. One or more of these skills, or none, may be on the actual exam.
2. Relative to the principal axes of the body (call the direction of the axes $\hat{x}$, $\hat{y}$, and $\hat{z}$, the principal moments of inertia $I_x$, $I_y$, and $I_z$. If the body is rotating with an angular velocity $\omega = -\omega_0\hat{x}$ (where $\omega_0$ is a positive number), what is the axis about which the rotation of this object occurs? [K] Relative to this axis, which way does this object rotate? [R] What angular momentum does this object have? [M]
3. At a certain instant, the angular momentum vector of an object relative to fixed cartesian coordinate system is given by:

   $$\vec{L} = (2\hat{x} + 7\hat{y} - 5\hat{z})\,\mathrm{kg\,m}^2/\mathrm{s}.$$

   while its angular velocity vector is given by:

   $$\vec{\omega} = (8\hat{x} + 4\hat{y})\,\mathrm{rad/s}.$$

   a. Is this system rotating about one of the body's principal axes? [E] Explain. [V]

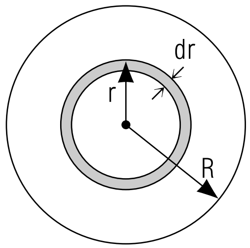
   b. What is the moment of inertia of this body with respect to the $x$-, $y$-, and $z$-axes? [H]

   c. What is the magnitude of $\vec{L}$? [P]

   d. What is the magnitude of $\omega$? [F]

   e. What is the moment of inertia of this body with respect to the axis determined by the direction of $\vec{\omega}$? [B]

4. A right circular cylinder of uniform volume density $\rho$, length $L$, and radius $R$, has a total mass $M$. It is shown in cross-section in the sketch (the length $L$ is perpendicular to the page).



a. Express the density $\rho$ in terms of $M$, $L$, and $R$. [J]

b. How much mass is in a cylindrical shell of thickness $dr$ (shown shaded in sketch), located at a distance $r$ from the axis of the cylinder? [T]

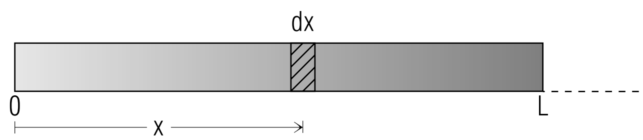
c. What is the moment of inertia, relative to the axis of the cylinder, of the mass contained in this cylindrical shell of thickness $dr$? [N]

d. Sum up, by integration, the contributions to the total moment of inertia from all of the cylindrical shells from the axis out to radius $R$. [U]

e. Eliminate $\rho$ from above to get $I$ in terms of $M$ and $R$. [A]

f. What is the radius of gyration of this cylinder relative to this axis? [S]

5.




A nonuniform rod of length $L$ has a density that increases linearly from the end at $x = 0$ to the end $x = L$. The density (mass per unit length) is given by $\rho = bx$, where $b$ is a constant and $x$ is the distance from the end. The density at the $x = 0$ end is zero.

a. At a distance $x$ from the $x = 0$ end, what is the mass of an increment of length $dx$? [G]

b. Sum up the contributions to the total mass from all of the elements of length $dx$ to find the total mass of the rod (in terms of $b$ and $L$). [O]

c. Where is the center of mass of this rod? If you can't recall the definition of this, refer to a text. [Q]

d. What is the moment of inertia, relative to an axis through $x = 0$, of the element of mass located in thickness $dx$ at distance $x$ from the $x = 0$ end? [X]

e. What is the total moment of inertia of this rod relative to an axis through the $x = 0$ end? [L] What is the radius of gyration of this rod relative to the $x = 0$ end? [Y]





f. What is the moment of inertia relative to an axis through the center of mass? (Use the parallel axis theorem.) [I]

g. What is the moment of inertia relative to an axis through the $x = L$ end? (Get the answer using the parallel axis theorem and by direct integration.) [W]

**Brief Answers**:

A. $MR^2/2$.

B. $0.55\,\mathrm{kg\,m^2}$.

E. No.

F. $\sqrt{80}\,\mathrm{rad/s} = 8.94\,\mathrm{rad/sec}$.

G. $bx\,dx$.

H. Cannot be determined from information given.

I. $I_c = ML^2/18$.

J. $M/(\pi r^2 L) - 1$.

K. The $x$-axis.

L. $ML^2/2$.

M. $I_x\omega_0$, in the same direction as $\vec{\omega}$.

N. $2\pi\rho L r^3\,dr$.

O. $M = -bL^2/2$, so $b = 2M/L^2$.

P. $\sqrt{78}\,\mathrm{kg\,m^2/s}$.

Q. $X_c = 2L/3$.

R. Use the right-hand rule: if thumb is in direction of $\omega$, sense of rotation is in direction of curled fingers.

S. $\sqrt{0.5R^2} = 0.707\,R$.

T. $2\rho L\pi r dr$.

U. $\frac{1}{2}\pi\rho L^4$.

V. Because $\vec{L}$ and $\vec{\omega}$ are not in the same direction.

W. $I = 13/8ML2$.

X. $dI = bx^3\,dx$.

Y. $k = 0.707L$.