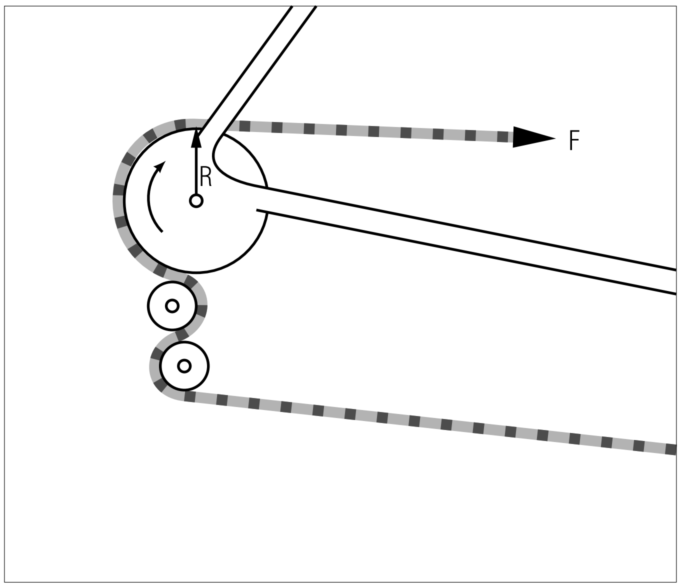
MISN-0-33

ANGULAR ACCELERATION IN CIRCULAR MOTION



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ANGULAR ACCELERATION IN CIRCULAR MOTION

by

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Title: **Angular Acceleration in Circular Motion**

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Input Skills:

1. Given a force at right angles to an axis, calculate the torque it produces about that axis (MISN-0-41).
2. Calculate the moment of inertia, of a given planar set of point masses, with respect to a perpendicular axis (MISN-0-41).
3. State the relationship between the angular velocity of a point and its radius and tangential velocity (MISN-0-9).
4. State Newton's second law (MISN-0-15).

Output Skills (Knowledge):

K1. For circular motion, differentiate $s = \theta r$ to derive the relation between linear acceleration, angular acceleration, and radius.

K2. Starting from Newton's second law, derive the relation between torque, moment of inertia, and angular acceleration for a point mass in circular motion.

K3. For circular motion with constant angular acceleration, derive the general expressions for angular velocity and angular acceleration as functions of time (involves integrals). Check the answers by differentiation.

K4. For circular motion with constant angular acceleration, eliminate the time variable between the angular displacement and angular velocity expressions to obtain angular velocity as a function of angular displacement and angular acceleration.

Output Skills (Problem Solving):

S1. For circular motion with constant angular acceleration, solve problems involving torque, moment of inertia, angular velocity, rotational displacement, and time.

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1. Two Agents of Change

In some instances of circular motion we observe only uniformity. The motion of planets around the sun or of the moon around the earth are examples. In many other cases, however, the rate of motion speeds up or slows down with time. A spinning figure skater, a flywheel, or the wheel of a bicycle will usually have an increasing or decreasing speed of rotation. One cause of change in the speed of rotation is a change in the moment of inertia of the rotating body. An example is arm extension by a spinning ice skater. Another cause of change in circular motion is an applied external torque. For example, a bicycle's wheel will increase its rotational speed due to torque exerted through the pressure of the bicyclist's foot on the pedal. In this module we will examine the relationship between applied torque and circular motion.

2. Angular and Linear Acceleration

2a. Angular Acceleration: $\vec{\alpha} = d\vec{\omega}/dt$. When the angular velocity of a point moving with circular motion is not constant, we describe its rate of change by introducing an angular acceleration in a manner analogous to linear acceleration. Recall that acceleration in linear motion is defined as the rate of change of velocity with time:

$$\vec{a} = \frac{d\vec{v}}{dt}. \tag{1}$$

In a similar manner, the angular acceleration[1] is defined as the rate of change of angular velocity with time:

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}. \tag{2}$$

[1] The Greek letter α, pronounced "al′ fa," is universally used to denote angular acceleration.

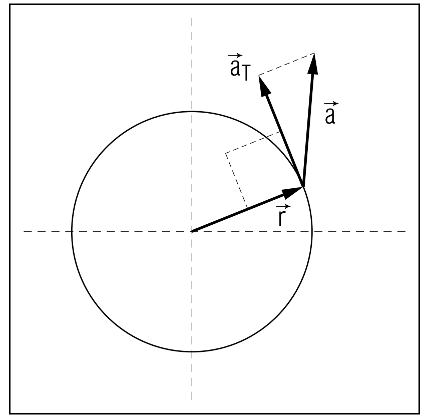

Figure 1. The tangential component $\vec{a}_T$, of acceleration $\vec{a}$, is that component perpendicular to $\vec{r}$.

2b. Relationship of α to a_T. For the case of circular motion we can avoid the complexities of derivatives of vectors by relating the magnitude of the tangential acceleration to the magnitude of the angular acceleration. The tangential component of the acceleration is defined as that component which is perpendicular to the radius vector, as shown in Fig. 1.

For circular motion, velocity is related to angular velocity by:[2]

$$v = \omega r. \qquad \text{(circular motion)} \tag{3}$$

This velocity is entirely tangential ($\vec{v}_T = \vec{v}$). Then taking the time derivative of Eq. (3) and using the chain rule:

$$a_T = \frac{dv}{dt} = r\frac{d\omega}{dt} + \omega\frac{dr}{dt} = r\frac{d\omega}{dt} \qquad \text{(circular motion)}\,.$$

Thus:

$$a_T = r\alpha \qquad \text{(circular motion)}. \tag{4}$$

Note that in these equations the units of ω must be radians per unit time and the units of α must be radians per unit time squared.

2c. Relationship of $\vec{\alpha}$ to $\vec{a}$. For a point that is $\vec{r}$ away from a circular rotator's center of rotation, the point's angular acceleration can be calculated from its linear acceleration by the relation:

$$\vec{\alpha} = (\vec{r} \times \vec{a})/r^2; \qquad \text{(rigid rotator)}\,.$$

[2] See "Kinematics: Circular Motion" (MISN-0-9) for a derivation of Equation (3).

This can be derived by taking the time derivative of the definition of angular velocity,[3]

$$\vec{\omega} = (\vec{r} \times \vec{v})/r^2\,,$$

and then making use of: $\vec{v} \times \vec{v} = 0$, $\vec{\alpha} \equiv d\vec{\omega}/dt$ and, for a point on a rigid rotator, $dr/dt = 0$. *Help: [S-9]*

3. Producing Angular Acceleration

3a. $\tau = I\alpha$. For a point mass in circular motion, for example on the rim of a flywheel at radius r, we can easily relate torque to angular acceleration. We start with the mass's tangential acceleration [see Equation (4)]:

$$a_T = \alpha r\,.$$

By Newton's second law, the tangential force F_T is:

$$F_T = mr\alpha. \qquad \textit{Help: [S-1]} \tag{5}$$

so the magnitude of the torque $\vec{\tau}$ is:[4]

$$\tau = rF_T = I\alpha\,. \tag{6}$$

where $I \equiv mr^2$ is the moment of inertia of that mass about the rotation axis. Although it will not be shown here, the fact that α is the same for all points in a rigidly rotating object allows one to prove that:[5]

$$\tau_{\text{ext}} = I\alpha. \tag{7}$$

for the object as a whole. Here I is the total moment of inertia of the object about the axis, obtained by summing the values of I for the object's component masses, and τ_{ext} is the net external torque about the axis.

3b. Deriving $\vec{\tau} = I\vec{\alpha}$. For a general object, angular acceleration is related to torque by a vector equation. Consider a mass m in a rigid rotator at position $\vec{r}$ from the center of rotation. If force $\vec{F}$ acts on this

[3] The more usual, but equivalent, definition of angular velocity is $\omega \equiv d\theta/dt$ and $\hat{\omega} \equiv (\vec{r} \times \vec{v})/|\vec{r} \times \vec{v}|$. Thus $\vec{\omega}$ is perpendicular to the plane defined by $\vec{r}$ and $\vec{v}$. The quantity $\vec{\omega}$ cannot be defined by a single equation: for example $\vec{\omega} \neq d\theta/dt$ because θ is not a vector quantity.

[4] In vector notation, Eq. (6) is written: $\tau = rF_\perp$, where $F_\perp$ is the component of $\vec{F}$ perpendicular to the radius vector $\vec{r}$.

[5] See "Torque and Angular Momentum in Circular Motion" (MISN-0-34).

mass, we can take the cross product of $\vec{r}$ with both sides of Newton's second law to get:

$$\vec{r} \times \vec{F} = m\vec{r} \times \vec{a}\,.$$

Eliminating $\vec{a}$ for $\vec{\alpha}$ and using $I \equiv mr^2$ and $\vec{\tau} \equiv \vec{r} \times \vec{F}$, get:

$$\vec{\tau} = I\vec{\alpha}\,.$$

The same relation could be obtained by integrating over all the mass of an extended object, with $\vec{\tau}$ becoming the external torque on the entire object.[6]

Of course the torque, moment of inertia and angular acceleration are all to be computed with respect to the system's designated center of rotation.

4. Constant Torque Case

4a. Angular Kinematics. The simplest applications of $\tau = I\alpha$ will be those in which all three quantities τ, I, and hence α, are constant. Just as in the case of linear motion,[7] one can then easily integrate the equations relating θ, ω, and α to obtain those quantities as functions of time. First we integrate $d\omega/dt = \alpha$, assuming constant α, to obtain:

$$\omega = \omega_0 + \alpha t; \qquad (\text{constant } \alpha). \tag{8}$$

Next we integrate $d\theta/dt \equiv \omega$ (see above) to obtain:

$$\theta = \theta_0 + \omega_0 t + \alpha t^2/2; \qquad (\text{constant } \alpha). \tag{9}$$

Eqs. (8) and (9) can easily be checked by differentiation plus evaluation at $t = 0$.

4b. From a Dead Start, $\omega = (2\alpha\theta)^{1/2}$. We can eliminate the time variable between Eqs. (7) and (8) for the case $\theta_0 = \omega_0 = 0$ and obtain the useful relation:

$$\omega = (2\alpha\theta)^{1/2}; \qquad (\text{from a dead start, constant } \alpha)\,.$$

[6] When $\vec{\tau}$ and $\vec{\alpha}$ are not in the same direction, I is a tensor quantity.

[7] See "Kinematics: Motion in One Dimension," (MISN-0-7).

4c. Constant-α and Constant-a Equations. Most of us find it easy to remember the circular-motion constant-α equations by analogy with the linear-motion constant-a equations:

Kinematics Magnitudes		
Position	Velocity	Dead-Start Velocity
$s = s_0 + v_0 t + at^2/2$	$v = v_0 + at$	$v = (2as)^{1/2}$
$\theta = \theta_0 + \omega_0 t + \alpha t^2/2$	$\omega = \omega_0 + \alpha t$	$\omega = (2\alpha\theta)^{1/2}$

Force/Torque, Energy, Power Magnitudes		
Kinetic energy	Force/Torque	Constant-Velocity Power
$E_k = mv^2/2$	$F = m\,a$	$P = Fv$
$E_k = I\omega^2/2$	$\tau = I\alpha$	$P = \tau\omega$

Some of the equations shown in these two tables can be rewritten in vector form so they have validity beyond linear and circular motion.

Acknowledgments

We wish to thank Bill Lane, Steve Smith and their students for suggestions that have greatly improved this module. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

PROBLEM SUPPLEMENT

1. The angular displacement of a particle moving in a circular path of radius R is given as a function of time by:

$$\theta(t) = at^2 + bt + c,$$

where a, b, and c are constants; θ is to be expressed in radians and t in seconds.

a. What are the units of the constant a? *Answer: 7*

b. What are the units of the constant b? *Answer: 4*

c. What is the angular velocity of the particle at time t? *Answer: 6*

d. What is the particle's angular acceleration at time t? *Answer: 2*

e. What is its (tangential) velocity at time t? *Answer: 5*

f. Is this tangential speed constant? *Answer: 1*

g. What is the tangential acceleration of this particle at time t? *Answer: 3*

h. Is this tangential acceleration constant? *Answer: 8*

2. A Volvo marine engine goes from 1800 rpm to 5400 rpm in 24 seconds. Assume the engine's moment of inertia is 100 ft lb s^2. What average torque is developed by the engine? *Answer: 9 Help: [S-2]*

3. Assume the mass of a coasting flywheel is 140 kg, essentially all located at the rim at a radius of 1.2 m from the axis of rotation. The bearings are located at a radius of 4 cm and the initial angular velocity is 6 rad/s. By the end of one minute, friction has decreased the angular velocity to 5.4 rad/s.

a. The moment of inertia of the flywheel. *Answer: 12*

b. The flywheel's average angular acceleration. *Answer: 10 Help: [S-4]*

c. The average frictional torque exerted by the bearings on the flywheel. *Answer: 13 Help: [S-5]*

d. The total average frictional force exerted by the bearings on the flywheel. *Answer: 11 Help: [S-6]*

4. The flywheel of a miniature steam engine has a mass of 0.20 kg, located almost entirely along the rim at a radius of 8.0 cm. While the flywheel is rotating at 120 rpm it is decoupled from the engine and a braking force of 0.050 N is applied to the rim until the flywheel stops. For the time period during which the braking occurs, determine the flywheel's:

   a. moment of inertia *Answer: 16*

   b. angular acceleration *Answer: 18*

   c. tangential acceleration of a point on the rim *Answer: 15*

   d. total braking time *Answer: 17 Help: [S-7]*

   e. total number of turns (a "turn" is one complete rotation) *Answer: 14 Help: [S-8]*

Brief Answers:

1. No
2. $2\,a$
3. $2\,aR$
4. rad/s
5. $(2at + b)\,R$
6. $2at + b$
7. rad/s^2
8. Yes
9. 1570 ft lb
10. -0.01 rad/s^2
11. 50.4 N, opposite to direction of flywheel velocity at bearings.
12. 201.6 kg m^2
13. 2.016 N m, opposite to direction of flywheel angular velocity vector.
14. 4.02 turns

15. -0.25 m/s^2
16. 1.28×10^{-3} kg m^2
17. 4.02 s
18. -3.13 rad/s^2

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-3a)*

Note that the vectors $\vec{r}$ and $\vec{F}_T$ are at right angles so that θ is 90° in the expression: $|\vec{r} \times \vec{F}_T| = rF_T \sin\theta$.

S-2 *(from PS, problem 2)*

$\tau = I\,\alpha$. Therefore, averaging over both sides (I is constant): $\bar{\tau} = I\,\bar{\alpha}$. Also: $\bar{\alpha} = \dfrac{\Delta\omega}{\Delta t}$. *Further Help: [S-3]*

S-3 *(from [S-2])*

What are the units of angular velocity?
What are the units of frequency?
Which one is given in the problem?
Which one occurs in the equation you are using?
What is the conversion factor between them?
If you are unable to answer any of the above questions, see the Input Skills reference in this module's *ID Sheet* or see the Volume's *Index*.

S-4 *(from PS, problem 3b)*

Do Problem 2 completely and successfully before attempting this one.

S-5 *(from PS, problem 3c)*

First calculate the torque exerted on the flywheel, without considering where it is coming from.

S-6 *(from PS, problem 3d)*

The torque is exerted by the bearings at their surface of contact with the flywheel.

S-7 *(from PS, problem 4d)*

$\Delta\omega = \alpha\,\Delta t$. Also, see [S-2].

S-8 *(from PS, problem 4e)*

Under constant angular acceleration, $\theta = (1/2)\alpha t^2$. How many radians are there in a "turn"?

S-9 *(from TX-2c)*

We take the time-derivative of both sides of the beginning equation and that yields the final equation. Here are the details.
(1) For the left-hand side of the equation:
Expand the left side, the angular velocity, in Cartesian Coordinates: $\vec{\omega} = \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}$.
Take the time-derivative of both sides, using Eq. (3) of MISN-0-1, and using the fact that the Cartesian unit vectors are time-independent ($d\hat{x}/dt = 0$, etc.). Also make use of $\alpha_x \equiv d\omega_x/dt$, etc. Since $\alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} = \vec{\alpha}$, this results in α as the answer for the left side of the equation.
(2) For the right-hand side of the equation (and putting in our new left side):

$$\vec{\alpha} = \frac{d}{dt}\left(\frac{\vec{r}\times\vec{v}}{r^2}\right)$$

$$= \vec{\alpha} = \frac{d(\vec{r}\times\vec{v})}{dt}\left(r^{-2}\right) + (\vec{r}\times\vec{v})\frac{d}{dt}(r^{-2}).$$

Now r is independent of time for our constant-radius motion so the last term is zero and:

$$\vec{\alpha} = \frac{d(\vec{r}\times\vec{v})}{dt}\left(r^{-2}\right)$$

$$= \left(\frac{d\vec{r}}{dt}\times\vec{v}\right)\left(r^{-2}\right) + \left(\vec{r}\times\frac{d\vec{v}}{dt}\right)\left(r^{-2}\right).$$

The first term on the right side is $(\vec{v}\times\vec{v}/r^2)$ which is zero because the cross product of any vector with itself is zero [see MISN-0-2, Eq. (4), for the case $\theta = \phi$]. The second term is $(\vec{r}\times\vec{a})/r^2$. Then:
$\alpha = (\vec{r}\times\vec{a})\left(r^{-2}\right)$.

MODEL EXAM

1. See Output Skills K1-K4 on this module's *ID Sheet*.

2. The flywheel of a miniature steam engine has a mass of 0.20 kg, located almost entirely along the rim at a radius of 8.0 cm. While the flywheel is rotating at 120 rpm it is decoupled from the engine and a braking force of 0.050 N is applied to the rim until the flywheel stops. For the time period during which the braking occurs, determine the flywheel's:

   a. moment of inertia

   b. angular acceleration

   c. tangential acceleration of a point on the rim

   d. total braking time

   e. total number of turns

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 4.