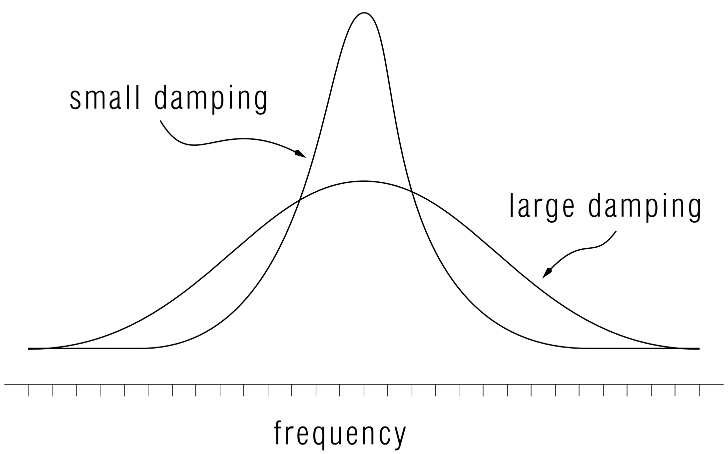
PHYSNET

MISN-0-31

DAMPED DRIVEN OSCILLATIONS; MECHANICAL RESONANCES



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

DAMPED DRIVEN OSCILLATIONS; MECHANICAL RESONANCES

by
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Title: **Damped Driven Oscillations; Mechanical Resonances**

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Input Skills:

1. Show: $x(t) = A\,e^{-\gamma t}\sin(\omega t + \alpha)$ solves $F = -kx - \lambda v$ (MISN-0-29).

Output Skills (Knowledge):

K1. Given the damped driven oscillator force, $F = -kx - \lambda v + F_0\cos(\omega t)$, and the steady-state solution, $x(t) = B(\omega) + \sin[\omega t + \beta(\omega)]$, derive expressions for $B(\omega)$ and $\beta(\omega)$.

K2. Show that $x(t) = x_T(t) + x_S(t)$ is a solution to the damped, driven oscillator problem, where the "transient solution" $x_T(t)$ is known to be a solution to the undriven case, and x_S is the steady-state solution.

K3. Derive the time-average steady-state power transferred into a damped driven oscillator (from a cosine driving force): $P_{ave}(\omega) = (F_0^2\lambda\omega^2/2)\,[m^2(\omega_0^2 - \omega^2)^2 + \lambda^2\omega^2]^{-1}$.

K4. Sketch P_{ave} vs. ω in the vicinity of the resonant frequency of a damped driven oscillator, both for a broad resonance and for a narrow one. Label each curve as to relative size of damping constant.

External Resources (Optional):

1. You might like to see an alternative presentation in some General Physics textbook (for availability, see this module's *Local Guide*).

DAMPED DRIVEN OSCILLATIONS; MECHANICAL RESONANCES

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1. Introduction

The concept of "resonance" is one of the most important in all of science and technology. There are many processes where resonance is essential: other processes where it is to be avoided at all cost. Included in the former category are the parts of instruments which produce music; in the latter, the recording and playback apparatus. In the former, the vocal cords of a person speaking or a bird singing; in the latter, most parts of the ear of a person or a bird listening. Here we assume the existence of a Hooke's Law type single-frequency driving force and examine the power transfer relationships in a realistic model of driven devices.

2. Transient & Steady-State Equations

2a. Combining $F = ma$ and the Oscillator Force. We combine Newton's second law, $F = ma$, with the damped driven oscillator force,

$$F = kx - \lambda v + F_0 \cos \omega t,$$

to obtain the basic equation for the damped driven oscillator:

$$mx'' + \lambda x' + kx - F_0 \cos \omega t = 0 \tag{1}$$

where primes indicate derivatives with respect to time.

2b. Steady-State/Transient Separation. The unique solution to this equation is[1]

$$x(t) = x_t(t) + x_s(t) \tag{2}$$

where the transient part is

$$x_t(t) = Ae^{-\gamma t} \sin(\omega_1 t + \alpha)$$

and the steady state part is

$$x_s(t) = B(\omega) \sin[\omega t + \beta(\omega)]$$

[1]See Appendix A.

The amplitude A and phase α of the transient part can be specified by, for example, displacement and velocity at time zero. However, the amplitude B and phase β of the steady state part are independent of such conditions and remain to be found.

2c. The Transient Solution. If Eq. (2) is substituted into Eq. (1) and x_t terms are separated from x_s terms, one gets:

$$(mx_t'' + \lambda x_t' + kx_t) + (mx_s'' + \lambda x_s' + kx_s - F_0 \cos \omega t) = 0 \tag{3}$$

Now it can be demonstrated[2] that

$$mx_t'' + \lambda x_t' + kx_t = 0 \tag{4}$$

if

$$x_t(t) = Ae^{-\gamma t} \sin(\omega_1 t + \alpha)$$

where

$$\gamma \equiv \lambda/2m$$

$$\omega_1 \equiv \sqrt{\omega^2 - \gamma^2}$$

$$\omega_0 \equiv \sqrt{k/m}$$

and A and α are still undetermined.

2d. The Steady State Equation. Without further ado we assume Eq. (4) and write Eq. (3) as:

$$mx_s'' + \lambda x_s' + kx_s - F_0 \cos \omega t = 0 \tag{5}$$

We offer two methods for the solution of this equation.

3. Solving the Steady-State Equation

3a. Algebraic Method. Substitute $x_s(t) = B \sin(\omega t + \beta)$ into Eq. (5) and get:

$$(\omega_0^2 - \omega^2)mB \sin(\omega t + \beta) + \lambda\, \omega B \cos(\omega t + \beta) - F_0 \cos(\omega t) = 0 \tag{6}$$

Now use the identity

$$\cos(A - B) = \cos A \cos B + \sin A \sin B,$$

[2]See "Damped Mechanical Oscillations" (MISN-0-29).

which should be on instant recall, with $A = \omega t + \beta$ and $B = \beta$ to get

$$\cos \omega t = (\sin \beta) \sin(\omega t + \beta) + (\cos \beta) \cos(\omega t + \beta).$$

Put that into Eq. (6) and collect terms:

$$[(\omega_0^2 - \omega^2)Bm - F_0 \sin \beta] \sin(\omega t + \beta) + (\lambda\, \omega B - F_0 \cos \beta) \cos(\omega t + \beta) = 0$$

Since the sine and cosine are independent functions of time, each of their constant coefficients must separately be zero in order to make the sum of the terms stay zero at all times.[3] Then:

$$(\omega_0^2 - \omega^2)Bm = F_0 \sin \beta \qquad (7)$$

$$\lambda\, \omega B = F_0 \cos \beta$$

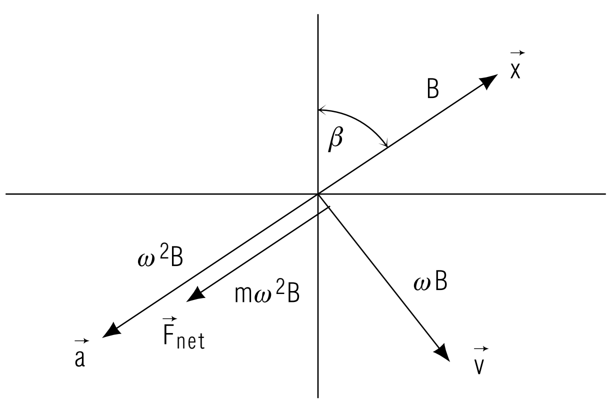
Dividing these two equations gives $\tan \beta$ and summing their squares gives B.

3b. Phasor Method. [4] First, since $x_s(t) = B \sin(\omega t + \beta)$,

$$F_s = m a_s = m x_s'' = -\omega^2 m x_s$$

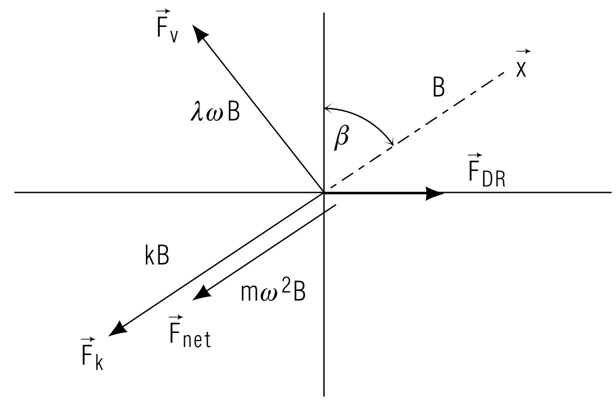
$$F_s = -k x_s \qquad \text{where } k \equiv \omega^2 m$$

so we can draw a phasor diagram for $t = 0$:



Now draw a force-phasor diagram for $t = 0$, noting that each force phasor is a constant times one of the phasors in the above diagram. At $t = 0$,

[3]See "Damped Mechanical Oscillations" (MISN-0-29).
[4]See " SHM Phasors" (MISN-0-27).



where $\vec{F}_k = -k\vec{x}_s$ (opposite to $\vec{x}_s$)

and $\vec{F}_v = -\lambda \vec{v}_s$ (opposite to $\vec{v}_s$).

Using

$$\vec{F}_{net} = \vec{F}_k + \vec{F}_v + \vec{F}_{DR}$$

we equate components $\parallel$ and $\perp$ to $\vec{F}_{net}$ and obtain Eqs. (7).

4. Average Power Dissipation

4a. Setting Up the Time-Average Integral. The time-average steady-state power transferred into the oscillator from the driving force can be determined via the power-force-velocity relation:[5]

$$P(t) = F(t) v(t)$$

Then over one period P the average power is:

$$P_{ave} = \frac{1}{P} \int_0^P P(t)\, dt = \frac{1}{P} \int_0^P F_{DR}(t) v(t)\, dt$$

where the driving force is used because we are trying to obtain the average power expended by that force. Substituting $F_{DR}(t)$ and $v(t)$:

$$P_{ave} = \frac{1}{P} \int_0^P F_0 (\cos \omega t) B \omega \cos(\omega t + \beta)\, dt.$$

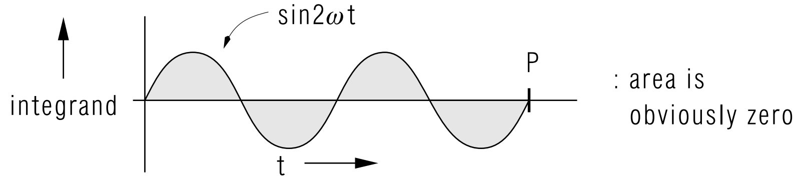
[5]See "Work,Kinetic Energy, Power" (MISN-0-20.

Again use: $\cos(A+B) = \cos A \cos B - \sin A \sin B$, now with $A = \omega t$ and $B = \beta$:

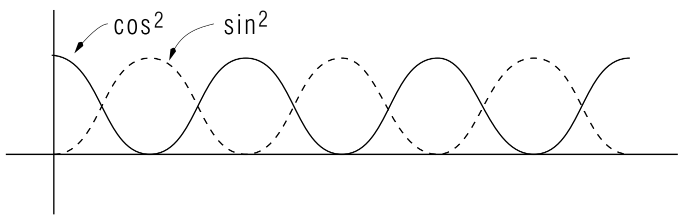
$$P_{ave} = \frac{F_0\,\omega}{P}\left[\int_0^P \cos^2 \omega t\, dt - \sin\beta \int_0^P \cos\omega t \sin\omega t\, dt\right]$$

4b. Integrating. The second integral is zero, as can be easily seen by writing it as:

$$\int_0^P \frac{1}{2}(\sin 2\omega t)\, dt$$



The value of the first integral is $P/2$ because the average value of $\cos^2$ is $1/2$. This is easily seen by making use of the fact that the average values of $\sin^2$ and $\cos^2$ are obviously equal:



Then:

$$\int_0^{2\pi} \cos^2 x\, dx = \int_0^{2\pi} sin^2 x\, dx$$

which can be used to evaluate the average value of $\cos^2 x$:

$$\begin{aligned}(\cos^2 x)_{ave} &= \frac{\int_0^{2\pi}\cos^2 x\,dx}{2\pi} = \frac{1}{2}\cdot\frac{\int_0^{2\pi}\cos^2 x\,dx}{2\pi} + \frac{1}{2}\cdot\frac{\int_0^{2\pi}\sin^2 x\,dx}{2\pi} \\ &= \frac{\int_0^{2\pi}(\cos^2 x + \sin^2 x)\,dx}{4\pi} = \frac{\int_0^{2\pi}(1)\,dx}{4\pi} = \frac{2\pi}{4\pi} = \frac{1}{2}.\end{aligned}$$

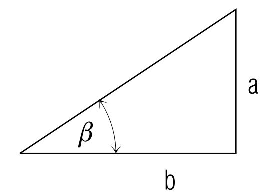
Then:

$$P_{ave} = \frac{F_0 B\omega\cos\beta}{2}$$

4c. Eliminating the Phase Angle. We can evaluate $\cos\beta$ from

$$\begin{aligned}\cos\beta &= \frac{1}{\sqrt{1+\tan^2\beta}} = \left[1 + \frac{m^2(\omega_0^2-\omega^2)^2}{\lambda^2\omega^2}\right]^{-1/2} \\ &= \frac{\lambda\omega}{[\lambda^2\omega^2 + m^2(\omega_0^2-\omega^2)^2]^{1/2}}\end{aligned}$$

However, rather than working out $\cos\beta$ from $\tan\beta$ as above, physicists often use a triangle. If $\tan\beta = a/b$ than a and b can be drawn as the legs of a right-angle triangle shown in the sketch at the right. The hypotenuse is obviously $\sqrt{a^2+b^2}$ and so $\cos\beta = b/\sqrt{a^2+b^2}$.



In our case:

$$\tan\beta = \frac{m(\omega_0^2-\omega^2)}{\lambda\omega}$$

hence

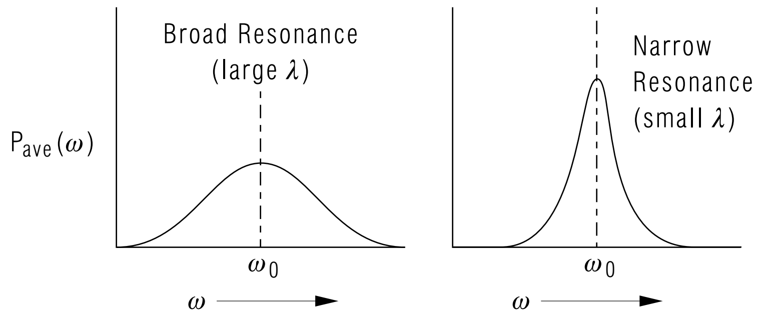
$$\cos\beta = (\lambda\omega)/\sqrt{\lambda^2\omega^2 + m^2(\omega_0^2-\omega^2)^2}$$

Using either derivation, the final answer is:

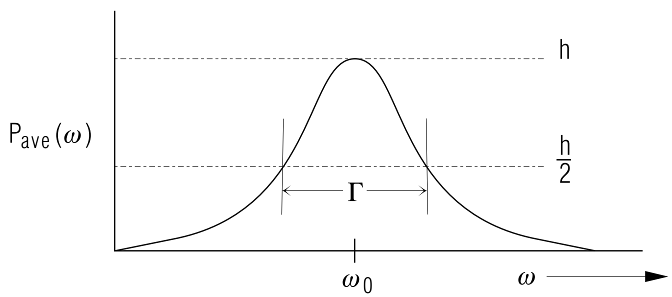
$$P_{ave} = \frac{F_0^2\,\omega^2\lambda}{2\,[m^2(\omega_0^2-\omega^2)^2 + \lambda^2\omega^2]}$$

5. Resonances

5a. Power Spectrums with Resonances. The resonant frequency is ω_0. If ω is swept through a range of frequencies, we get:



5b. Resonance Width (Approximate). It is easy to derive a formula for the width of the resonance in the "narrow width approximation." The width Γ of the resonance is defined at half the maximum resonance height h:



Then:

$$P_{ave}(\omega_{1/2}) = \frac{1}{2} P_{ave}(\omega_0)$$

Substituting in both sides:

$$\frac{F_0^2 \omega_{1/2}^2 \lambda}{m^2(\omega_0^2 - \omega_{1/2}^2)^2 + \lambda^2 \omega_{1/2}^2} = \frac{1}{2} \cdot \frac{F_0^2 \omega_0^2 \lambda}{\lambda^2 \omega_0^2},$$

which results in:

$$\omega_0^2 - \omega_{1/2}^2 = \pm \lambda\, \omega_{1/2}/m.$$

Factor the left hand side:

$$\omega_0^2 - \omega_{1/2}^2 = (\omega_0 - \omega_{1/2}) \cdot (\omega_0 + \omega_{1/2})$$

and use the narrow width approximation

$$\omega_0 \approx \omega_{1/2}$$

to get:

$$\omega_0 + \omega_{1/2} \approx 2\omega_0,$$
$$\omega_0 - \omega_{1/2} = \pm 2\lambda/2m,$$
$$\omega_{1/2} = \omega_0 \pm \lambda/2m.$$

The full width at half maximum is then:

$$\Gamma = \lambda/m.$$

This shows that the width is directly proportional to the damping strength. The average power put into the oscillator by the driving force is often written:

$$P_{ave}(\omega) = \frac{F_0^2 \Gamma/(8m)}{(\omega - \omega_0)^2 + (\Gamma/2)^2}$$

but you should realize that this is only valid in the narrow-width approximation. An interesting relation between the height and width of the resonance can be obtained by evaluating the height at ω_0:

$$P_{ave}(\omega_0) = \frac{F_0^2}{2\Gamma m},$$

so the height varies inversely as the width.

5c. What Happens as Damping Goes to Zero. A dramatic example of damping going to zero is a car wheel that is coming loose. As the damping constant fades, power gets increasingly fed in at the resonant frequency of vibration of the wheel until it is too much for the restraining system. Crack!

5d. Resonances as Complex-Plane Poles. A broad resonance is the shoulder of a far-away pole in the complex plane. As the damping constant decreases, the pole moves closer to the real axis. This causes the shoulder to narrow and heighten. In fact, the width of the resonance is the distance of the pole from the real axis. These matters are discussed and illustrated elsewhere.[6]

[6] See "Resonances and Poles: Relationship Between the Real and Imaginary Worlds" (MISN-O-49).

Acknowledgments

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A. Solving Inhomogeneous Differential Equations

There is a general approach to equations like (1), which are called inhomogeneous because not all terms contain the solution x (here, the final term omits it). The general solution is the sum of two terms: (a) the "homogeneous-part" solution that is adjusted to match boundary conditions; and (b) an entire-equation solution that has no adjustable parameters. In our case, the homogeneous part of Eq. (1) is Eq. (4): it's solution, following Eq. (4), shows the two adjustable constants. The unadjustable entire-equation solution, Eq. (7), obeys equation (5). The sum of these two "solutions" is our solution to the entire equation.

B. Resource Supplement

We suggest you look in General Physics textbooks (for availability, see this module's *Local Guide* under the subjects: (1) Forced Oscillations; and (2) Mechanical Resonances. There is usually at least a section on these topics. Skim the words or read them carefully, as is your pleasure, but we suggest you examine the illustrations carefully. In this subject it may be helpful to see other authors' presentations. You might also be interested in looking at the presentations in Mechanics textbooks such as Barger and Olsson, *Classical Mechanics*, Section 1-9, and Marion, *Classical Dynamics of Particles and Systems*, Section 4.2.

LOCAL GUIDE

You will find many General Physics textbooks in our Consulting Room bookcase.