PHYSNET

MISN-0-26
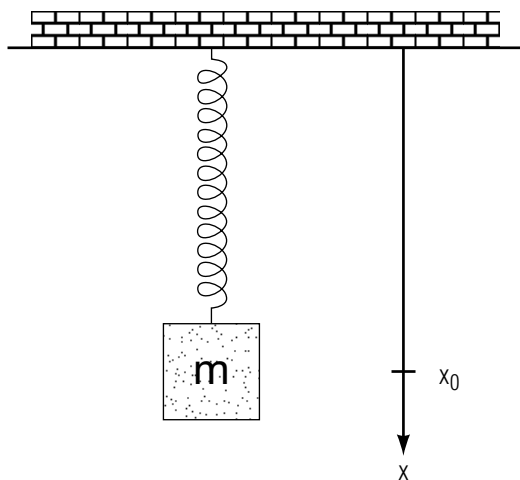
SIMPLE HARMONIC MOTION: SHIFTED ORIGIN AND PHASE

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by

Kirby Morgan

1. Dynamics of Harmonic Motion

2. The Force

3. Equation of Motion

Title: **Simple Harmonic Motion: Shifted Origin and Phase**

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Input Skills:

1. Vocabulary: potential energy curve (MISN-0-22), simple harmonic motion, displacement, equilibrium, amplitude, phase, initial time, angular frequency, period, frequency (MISN-0-25).
2. Calculate quantities such as displacement, velocity, force, angular frequency, and potential energy for SHM (MISN-0-25).
3. Find solutions of the equation: $d^2x/dt^2 + \omega^2 x = 0$.

Output Skills (Knowledge):

K1. Define: equilibrium position, displacement from equilibrium position, initial phase, restoring force, linear restoring force, Hooke's law force.

K2. State Hooke's law, identifying all quantities, and describe the limitations on its validity.

K3. Start with Newton's second law and Hooke's law and use them to obtain the equation of motion for a simple harmonic oscillator. Find its solutions and derive the relationship of frequency and period to force constant and mass.

K4. Define angular frequency, frequency and period in terms of the mass m and force constant k for a simple harmonic oscillator.

Output Skills (Problem Solving):

S1. Solve SHM problems involving an equilibrium position that is a finite distance from the origin, and with a finite initial phase. Solve for the force-constant, angular fequency, period, initial phase, and maximum acceleration in terms of given quantities, symbolically and numerically. Solve for the following functions of time as equations with symbols, numbers, and/or graphs as requested: displacement, velocity, acceleration.

SIMPLE HARMONIC MOTION: SHIFTED ORIGIN AND PHASE

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1. Dynamics of Harmonic Motion

1a. Force Varies in Magnitude and Direction. The force associated with simple harmonic motion[1] is probably a more complex kind of force than any you have encountered so far in dynamics. First you met forces of constant magnitude and direction. Then there was the centripetal force[2] which varied in direction but had a constant magnitude. Still another was the impulsive force,[3] whose direction was constant but whose magnitude varied. Here, the force associated with harmonic motion varies both in magnitude and direction.

1b. Potential Energy For a Restoring Force. The potential energy function E_p for a particle undergoing harmonic motion can be shown to have a minimum at its equilibrium position. Figure 1 shows the potential energy curve for a particle oscillating between the limits x_1 and x_2, with x_0 being the equilibrium position. The force acting on the particle is related to the potential energy by

$$F = -\frac{dE_p}{dx} \quad (1)$$

and thus is represented at any point by the negative value of the slope at that point. The slope is zero where the potential energy is minimum (at $x = x_0$), therefore the force is zero at the equilibrium position x_0. Moreover, since the slope is positive (negative) to the right (left) of x_0, the force always points towards x_0 *Help: [S-1]*,[4] making it a point of stable equilibrium.[1] This force is called a restoring force because it always acts to accelerate the particle back toward the equilibrium position.

1c. Potential Energy for SHM. The potential energy of a particle undergoing simple harmonic motion about the point x_0 is:

$$E_p = \frac{1}{2}k(x - x_0)^2. \quad (2)$$

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[1] See "Simple Harmonic Motion: Kinematics and Dynamics" (MISN-0-25).
[2] See "Centripetal and *g* Forces in Circular Motion" (MISN-0-17).
[3] See "Momentum, Force and Conservation of Energy" (MISN-0-15).
[4] For help, see sequence [S-1] in this module's *Special Assistance Supplement*.
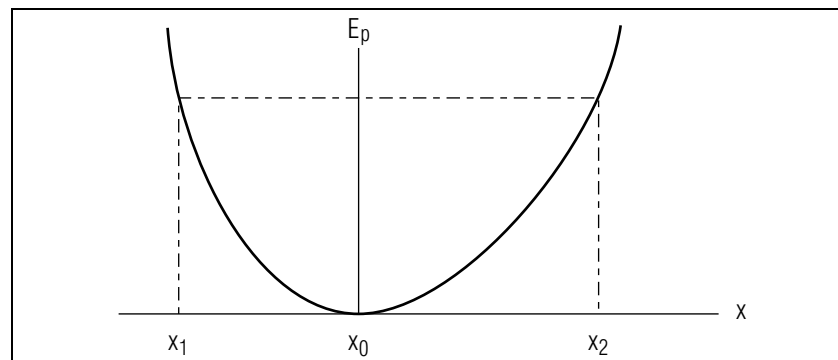
Figure 1. Potential Energy for a Restoring Force.

Given that the potential energy for SHM is $E_p = kx^2/2$ for displacement about the point $x = 0$,[1] Eq. (2) is the more general expression for motion about $x = x_0$. The potential energy curve for SHM (see Fig. 2) has a minimum at the equilibrium position and is symmetrical about that point. Thus the limits of the motion, the points where the potential energy equals the total energy, are equally spaced about the equilibrium position. The force is found from the potential energy:

$$F = -dE_p/dx$$
$$F = -k(x - x_0). \quad (3)$$

which is the general form of the SHM force.

2. The Force

2a. Hooke's Law. The relation $F = -k(x - x_0)$ was first found empirically by Robert Hooke in the 17th century and so is known as Hooke's

Figure 2. Potential energy for SHM.

Figure 3. Mass hung on a spring. Note that the position of stable equilibrium is at x_0, not at the origin.

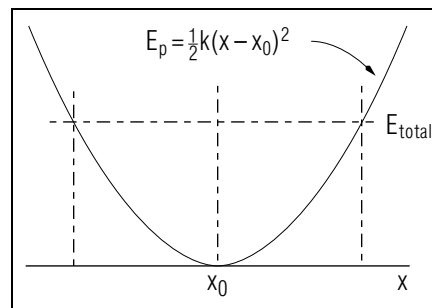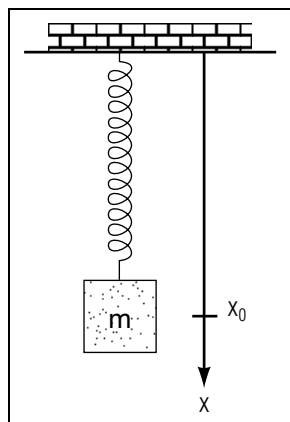law. Hooke found that, for springs and other elastic bodies, the restoring force was proportional to the displacement from equilibrium provided the elastic solid was not deformed beyond a certain point called the elastic limit. Past this point the elastic body will not return to its original shape when the applied force is removed. It turns out that Hooke's law holds for many common materials and can be applied to almost any situation where the displacement is small enough.[5] Since by Hooke's law the force is linearly proportional to the displacement, it is called a linear restoring force or, equivalently, a "Hooke's law force." The proportionality constant k is known as the "force constant."

2b. Example: Mass Hung on a Spring. Consider now a mass m attached to a spring of negligible mass and force constant k. The spring is hung vertically as shown in Fig. 3. The force acting to stretch the spring is just the weight of the mass, i.e. mg. When the mass is hung from the spring it causes the spring to stretch from its original length until the magnitude of the restoring force equals the weight:

$$mg = |F| = kx_0 \tag{4}$$

Here we have chosen x_0 as the distance the mass stretches the spring. The mass-spring-gravity system thus has an equilibrium point at $x = x_0$ and the resultant force is zero there. Now if the mass is displaced from x_0, Hooke's law states that it will experience a force $F = -k(x - x_0)$ acting so as to return the mass to x_0 (so long as the elastic limit of the spring is not exceeded).

[5]See "Small Oscillations" (MISN-0-28).

3. Equation of Motion

3a. Displacement Equation from Hooke's Law. Hooke's law, $F = -k(x - x_0)$, and Newton's second law, $F = ma$, can be combined to find the displacement equation for SHM. Writing the acceleration as d^2x/dt^2, the equation of motion is:

$$m\frac{d^2x}{dt^2} = -k(x - x_0), \tag{5}$$

or:

$$\frac{d^2x}{dt^2} + \frac{k}{m}(x - x_0) = 0. \tag{6}$$

Letting $\omega^2 \equiv k/m$ and $X \equiv x - x_0$, Eq. (6) can be rewritten in terms of X, the displacement from equilibrium:

$$\frac{d^2X}{dt^2} + \omega^2 X = 0 \tag{7}$$

which has the general solution[6]

$$X = a\cos\omega t + b\sin\omega t \tag{8}$$

or the totally equivalent general solution: *Help: [S-2]*

$$X = x(t) - x_0 = A\cos(\omega t + \delta_0). \tag{9}$$

This is just the displacement equation for a simple harmonic oscillator, where $X = x - x_0$ is the displacement, A is the amplitude, ω is the angular frequency, and δ_0 is the "initial phase."

3b. ω, ν and T in Terms of Force Constant k.. The angular frequency, ω, frequency, ν, and period, T, can all be expressed in terms of

[6]If you have not encountered this before, here is all you need to know about general solutions of equations like Eq. (7) for this module: If the highest derivative in the equation is a second derivative, and if all terms in the equation contain the unknown function on the right side of each term, then any solution that satisfies the equation is in fact the unique solution to the equation if the solution contains two independent constants undetermined by the equation. The two undetermined constants in Eq. (8) are a and b. The statement is very powerful, since it says that you need to know nothing about differential equations: just find a function with two constants that satisfies the equation (you must merely take derivatives) and you are guaranteed that you have the only solution! Equation (8) obviously satisfies the criterion. Note: If the highest derivative in such an equation is a first derivative, then only one undetermined constant is needed.

the force constant k and mass m. Since $\omega^2 = k/m$, the angular frequency is

$$\omega = (k/m)^{1/2}. \tag{10}$$

The period is found to be

$$T = \frac{2\pi}{\omega} = 2\pi(m/k)^{1/2}, \tag{11}$$

while the frequency is

$$\nu = \frac{1}{T} = \frac{1}{2\pi}(k/m)^{1/2}. \tag{12}$$

Observe that ω, ν, and T all depend only on k and m and are independent of the amplitude of the oscillations.

$\triangleright$ What would ω, ν, and T be when $m = 3\,\text{kg}$ and $k = 48\,\text{N/m}$? *Help: [S-3]*

3c. Mass Hung on a Spring. Suppose a spring of negligible mass and of force constant $9.8\,\text{N/m}$ hangs vertically and has a mass of $0.10\,\text{kg}$ attached to its free end. We define the direction of positive displacement x as being *downward*. The equilibrium point is found from Eq. (4) to be

$$x_0 = \frac{mg}{k} = \frac{(0.10\,\text{kg})(9.8\,\text{m/s}^2)}{9.8\,\text{N m}} = 0.10\,\text{m}$$

and Hooke's law can be written:

$$F = -(9.8\,\text{N/m})\,(x - 0.10\,\text{m}).$$

Suppose the mass is initially pulled up a distance $0.2\,\text{m}$ above its equilibrium position and then released. It will undergo SHM with its equation for displacement being

$$X(t) \equiv x(t) - x_0 = 0.20\,\text{m} \cos\left(\frac{9.9}{\text{s}}\,t + \pi\right)$$

where the angular frequency, $9.9\,\text{radians/s}$, was found from Eq. (10). *Help: [S-4]*[7] Note that $\delta_0 = \pi\,\text{radians} = 180°$ because the mass was released from the position of maximum negative displacement (hence zero velocity). *Help: [S-5]*

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[7]We normally omit the word "radians" in stating units; so when you see ω given in inverse seconds you should always take it to mean "radians per second."
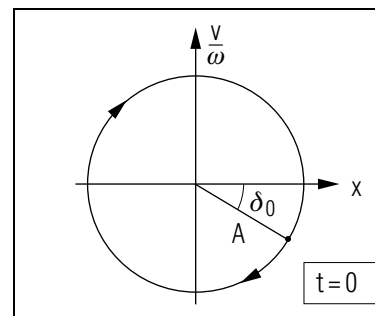
Figure 4. The oscillator point's position at time zero defines the oscillator's initial phase δ_0.

4. Scaled Phase Space

It is often useful to make a pictorial representation of the simultaneous values of an oscillator's position and velocity. If we scale the velocity axis by a factor of $(1/\omega)$, the point representing the values of the two variables traverses a circle as time progresses (see Fig. 4). In fact, it is easy to see that the point goes around the circle clockwise with the same frequency as the oscillator. Thus in the time it takes the oscillator to execute one complete cycle of its straight-line motion, the point representing it in the scaled phase space goes around its circle once. Then the "angular velocity" ω in the equation for the oscillator's linear displacement is just the angular velocity of the oscillator's point as it goes around its circle in the scaled phase space. Finally, on the scaled phase space diagram we can easily plot the oscillator's phase angle at time zero, its "initial phase," δ_0 (see Fig. 4).

5. Usefulness of Hooke's Law

Hooke's law, an empirical relation describing the elasticity of solids, has proved to be very helpful in the quantitative description of the SHM of such solids. Moreover the derivation of the equation of motion gives valuable insight into more complicated systems such as those where frictional forces are present.[8]

Acknowledgments

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[8]See "Damped Mechanical Oscillations" (MISN-0-29).

Glossary

- **elastic limit**: the point beyond which an elastic body does not obey Hooke's law.

- **equilibrium position**: the position where the force is zero so a particle put there would not move.

- **force constant**: the magnitude of the proportionality constant between force and displacement in Hooke's law.

- **Hooke's law force**: a restoring force that is linearly proportional to displacement from equilibrium (a "linear restoring force").

- **initial phase**: the phase of an oscillation at time zero.

- **linear restoring force**: a restoring force that is linearly proportional to displacement from equilibrium.

- **restoring force**: a force that always acts toward an equilibrium position.

PROBLEM SUPPLEMENT

Problem 8 also occurs in this module's *Model Exam*.

Note: Wherever appropriate, take the direction of **stretching** the spring as the direction of positive displacement.

1. One end of a horizontal spring is fixed, the other end is found to be moved 8.0 cm from its equilibrium position by a force of 20.0 N. A mass of 4.0 kg is then attached to that end of the spring and is pulled along a frictionless surface to a point 6.0 cm from the equilibrium position. The mass is then released at time $t = 0$ and it executes simple harmonic motion.

   a. Find the force constant of the spring.

   b. What is the force exerted by the spring on the mass just before it is released?

   c. Write the equation for the displacement of the mass as a function of time after it is released.

   d. Calculate the period and frequency of the oscillations.

2. A 3.0 kg mass hangs from a spring. A 0.50 kg body hung below the mass stretches the spring 5.0 cm farther. The body is then removed and the mass starts oscillating. Find the frequency of the motion.

3. A particle with mass 200.0 gm experiences a force whose associated potential energy function is $E_p = (4.0 \, \text{J/m}^2) \cdot (x - 0.20 \, \text{m})^2$. Find the angular frequency of the motion.

4. A 50.0 kg woman standing on the end of a diving board caused it to be lowered by 30.0 cm and it then oscillated up and down with her on it. Assuming the restoring force of the board was linear, calculate the number of times she went up and down in 2.2 s.

5. A car can be considered to be mounted on a vertical spring for purposes of describing its up and down motion. If the springs of a 1.5×10^3 kg car are adjusted so that the vibrations have a frequency of 3.0 Hz, what is the spring's force constant? Find the vibrational frequency if four passengers, averaging 75 kg each, ride in the car.

6. An object hung on a vertical spring stretches it 15 cm. The object is then pulled down another 20.0 cm and released.

   a. Find the displacement equation for the object.

   b. Find the velocity and acceleration as functions of time.

7. A vertically hanging spring has a force constant of 150.0 N/m. A 5.0 kg mass is now hung from the end of the spring and the mass settles down to a new equilibrium position x_0. Finally, the mass is displaced 0.20 m below its equilibrium position and given an initial velocity of 2.0 m/s downward.

   a. Write Hooke's law for the mass-spring system and calculate the displacement when the force is 30.0 N.

   b. Find the displacement equation.

   c. Calculate the period.

8. An object undergoing SHM has a mass of 2.0 kg and force constant 60.0 N/m. Its equilibrium position is at $x_0 = 0.30$ m.

   a. Write Hooke's law and find the position of the object when the force on it is 25 N.

   b. Find the displacement equation if the object: (1) has zero velocity when it is at the point $x = 0.80$ m; and (2) at time zero is at the equilibrium position and is heading toward negative displacements. Then find x at $t = 1.0$ s.

   c. Calculate the period of the oscillations.

Brief Answers:

1. a. $k = \dfrac{-F}{x - x_0} = -\dfrac{(-20 \text{ N})}{0.08 \text{ m}} = 250 \text{ N/m}$

   b. $F = -k(x - x_0) = -(250 \text{ N/m})(0.06 \text{ m}) = -15 \text{ N}$

   c. $\omega = \left(\dfrac{250 \text{ N/m}}{4.0 \text{ kg}}\right)^{1/2} = 7.9 \text{ rad/s}$

   $x(t) = 0.060 \text{ m} \cos\left(\dfrac{7.9}{\text{s}}t\right)$

   d. $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{7.9/\text{s}} = 0.80 \text{ s}; \ \nu = \dfrac{1}{T} = 1.3 \text{ Hz}$

2. $k = \dfrac{-F}{x - x_0} = \dfrac{mg}{x - x_0} = \dfrac{(0.50 \text{ kg})(9.8 \text{ m/s}^2)}{0.05 \text{ m}} = 98 \text{ N/m}$

   $\nu = \dfrac{1}{2\pi}\left(\dfrac{98 \text{ N/m}}{3.0 \text{ kg}}\right)^{1/2} = 0.91 \text{ Hz}$

3. $E_p = \dfrac{k}{2}(x - x_0)^2 = 4.0 \text{ J/m}^2 \, (x - x_0)^2$

   $\dfrac{k}{2} = 4.0 \text{ J/m}^2 \Rightarrow k = 8.0 \text{ N/m}$

   $\omega = \left(\dfrac{8.0 \text{ N/m}}{0.20 \text{ kg}}\right)^{1/2} = 6.3 \text{ rad/s}$

4. $k = \dfrac{-F}{x - x_0} = \dfrac{mg}{x - x_0} = \dfrac{(50 \text{ kg})(9.8 \text{ m/s}^2)}{0.30 \text{ m}} = 1600 \text{ N/m}$

   $T = 2\pi\left(\dfrac{50 \text{ kg}}{1600 \text{ N/m}}\right)^{1/2} = 1.1 \text{ s}$

   $\# = \dfrac{2.2 \text{ s}}{T} = \dfrac{2.2 \text{ s}}{1.1 \text{ s}} = 2 \text{ times}$

5. $k = m(2\pi\nu)^2 = (1500 \text{ kg})\left[2\pi(3.0 \text{ Hz})\right]^2 = 5.3 \times 10^5 \text{ N/m}$

   $\nu = \dfrac{1}{2\pi}\left(\dfrac{k}{m_{\text{total}}}\right)^{1/2} = \dfrac{1}{2\pi}\left(\dfrac{5.3 \times 10^5 \text{ N/m}}{1800 \text{ kg}}\right)^{1/2} = 2.7 \text{ Hz}$

6. $x_0 = \dfrac{mg}{k} \Rightarrow \dfrac{k}{m} = \dfrac{g}{x_0} = \dfrac{9.8 \text{ m/s}^2}{0.15 \text{ m}}$

   $\omega = \left(\dfrac{9.8 \text{ m/s}^2}{0.15 \text{ m}}\right)^{1/2} = 8.1 \text{ rad/s}$

   a. $x - 0.15 \text{ m} = 0.20 \text{ m} \cos\left(\dfrac{8.1}{\text{s}}t\right)$

   b. $v = \dfrac{dx}{dt} = (0.20 \text{ m})(8.1/\text{s})\left[-\sin\left(\dfrac{8.1}{\text{s}}t\right)\right]$

   $a = \dfrac{dv}{dt} = (-1.6 \text{ m/s})\left(\dfrac{8.1}{\text{s}}\right)\cos\left(\dfrac{8.1}{\text{s}}t\right) = -13 \text{ m/s}^2 \cos\left(\dfrac{8.1}{\text{s}}t\right).$

7. $x_0 = \dfrac{mg}{k} = \dfrac{(5.0\,\text{kg})(9.8\,\text{m/s}^2)}{150\,\text{N/m}} = 0.33\,\text{m}$

   a. $F = (-150\,\text{N/m})(x - 0.33\,\text{m})$

      displacement $= (x - 0.33\,\text{m}) = \dfrac{-30\,\text{N}}{-150\,\text{N/m}} = 0.20\,\text{m}$

   b. $\omega = \left(\dfrac{150\,\text{N/m}}{5.0\,\text{kg}}\right)^{1/2} = 5.5\,\text{rad/s}$

      $x - 0.33\,\text{m} = A\cos\left(\dfrac{5.5}{\text{s}}t + \delta_0\right)$

      $v = \dfrac{dx}{dt} = -A(\dfrac{5.5}{\text{s}})\sin\left(\dfrac{5.5}{\text{s}}t + \delta_0\right)$

      at $t = 0$: $0.20\,\text{m} = A\cos\delta_0$

      $2\,\text{m/s} = -A\left(\dfrac{5.5}{\text{s}}\right)(\sin\delta_0)$

      $(0.20\,\text{m})^2 + \left(\dfrac{2\,\text{m/s}}{5.5/\,\text{s}}\right)^2 = A^2(\cos^2\delta_0 + \sin^2\delta_0) = A^2$

      $A = 0.42\,\text{m}$

      $\delta_0 = \cos^{-1}\left(\dfrac{0.20\,\text{m}}{0.42\,\text{m}}\right) = 62° = 1.1\,\text{rad}$

      $x - 0.33\,\text{m} = 0.42\,\text{m}\cos\left(\dfrac{5.5}{\text{s}}t + 1.1\ \text{rad}\right)$

   c. $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{5.5/\,\text{s}} = 1.1\,\text{s}$

8. a. $F = (-60\,\text{N/m})(x - 0.30\,\text{m})$
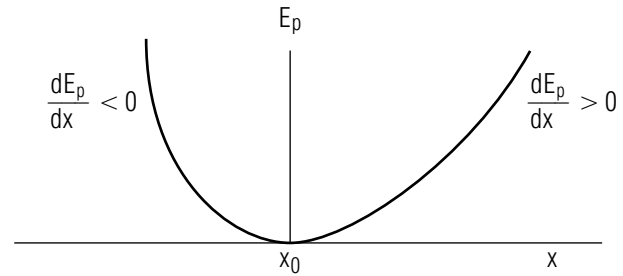
      $F = 25\,\text{N}$; $x = -0.12\,\text{m}$

   b. $x - 0.30\,\text{m} = 0.50\,\text{m}\cos\left(\dfrac{5.5}{\text{s}}t + \dfrac{\pi}{2}\right)$; $x(1.0\,\text{s}) = 0.65\,\text{m}$
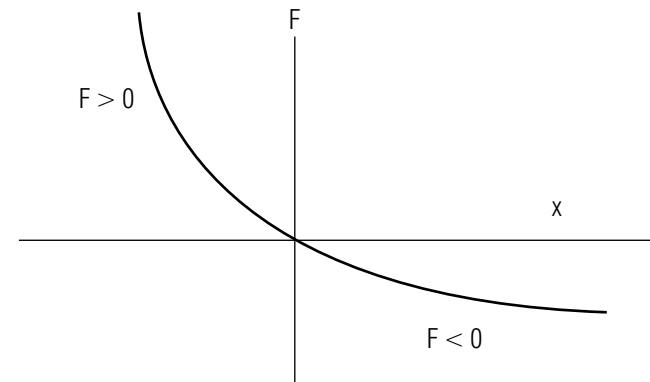
   c. $T = 1.1\,\text{s}$

SPECIAL ASSISTANCE SUPPLEMENT

S-1   *(from TX-1b)*

The force curve is related to the potential energy curve by $F = -dE_p/dx$:

Now draw the force curve for: $E_p = \dfrac{1}{2}k(x - x_0)^2$. *Help: [S-6]*

S-2   *(from TX-3a)*

$\cos(\theta + \phi) = \cos\phi\cos\theta - \sin\phi\sin\theta$

$X = A\cos(\omega t + \delta_0) = A[\cos\delta_0\cos\omega t - \sin\delta_0\sin\omega t]$

Let: $a \equiv A\cos\delta_0$; $b \equiv -A\sin\delta_0$.

Then: $X = a\cos\omega t + b\sin\omega t$.

S-3   *(from TX-3b)*

$\omega = \left(\dfrac{k}{m}\right)^{1/2} = \left(\dfrac{48\,\text{N/m}}{3\,\text{kg}}\right)^{1/2} = 4\,\text{rad/s}$

$T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{4\,\text{s}} = \dfrac{\pi}{2}\,\text{s}; \ \nu = \dfrac{1}{T} = \dfrac{1}{\pi/2\,\text{s}} = \dfrac{2}{\pi}\,\text{Hz}$

S-4   *(from TX-3c)*

$\omega = \left(\dfrac{k}{m}\right)^{1/2} = \left(\dfrac{9.8\,\text{N/m}}{0.1\,\text{kg}}\right)^{1/2} = 9.9\,\text{rad/s}$

S-5   *(from TX-3c)*

$x = A\cos(\omega t + \delta_0)$

$v = dx/dt = -A\omega\sin(\omega t + \delta_0)$

at $t = 0$; $v = -A\omega\sin\delta_0$

if $v = 0$ then $\delta_0 = 0$ (or $n\pi$); if $v \neq 0$ then $\delta_0 \neq 0$
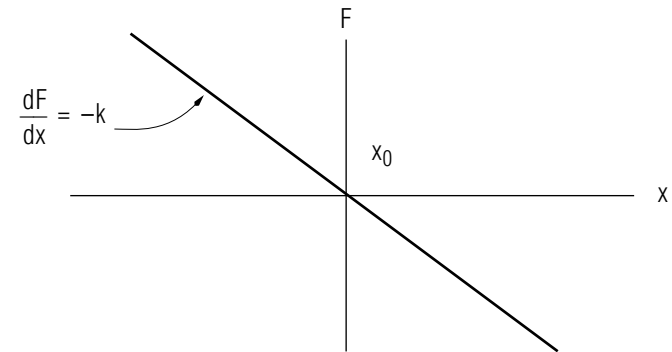
for $\delta_0 \neq 0$; $X_i = A\cos\delta_0$ $(t = 0)$

$|\cos\delta_0| < 1$: $X_i < A$

i.e. amplitude is larger than initial displacement X_i.

S-6   *(from [S-1])*

$F = -dE_p/dx = -k(x - x_0)$

MODEL EXAM

1. See Output Skills K1-K4 in this module's *ID Sheet*.

2. An object undergoing SHM has a mass of $2.0\,\text{kg}$ and force constant $60.0\,\text{N/m}$. Its equilibrium position is at $x_0 = 0.30\,\text{m}$.

   a. Write Hooke's law and find the position of the object when the force on it is $25\,\text{N}$.

   b. Find the displacement equation if the object has zero velocity at the point $x = 0.80\,\text{m}$. Find x at $t = 1.0\,\text{s}$.

   c. Calculate the period of the oscillations.

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 8.