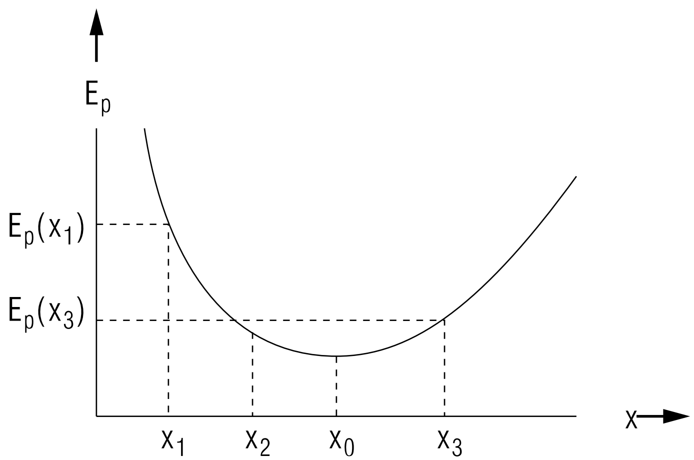
PHYSNET

MISN-0-22

ENERGY GRAPHS, MOTION, TURNING POINTS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

ENERGY GRAPHS, MOTION, TURNING POINTS

by

J. S. Kovacs, Michigan State University

Title: **Energy Graphs, Motion, Turning Points**

Author: J. S. Kovacs, Physics Dept., Mich. State Univ., E. Lansing, MI

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Input Skills:

1. State the relationship between the line integral of a conservative force between specified limits and the potential energy difference between those limits (MISN-0-21).
2. Given the graph of a simple function, such as $f = x \exp(-a x)$, demonstrate the relationship between slope and derivative at any given point (MISN-0-1).
3. State the law of conservation of energy for mechanical systems, defining kinetic and potential energy (MISN-0-21).

Output Skills (Knowledge):

K1. Vocabulary: potential energy curve, energy diagram, turning point.

Output Skills (Problem Solving):

S1. Given the potential energy of a particle as a function of position (in one dimension or radially with spherical symmetry) determine (for a given position) the force acting on that particle and its acceleration, velocity, and turning points (if any).

S2. Given the graph of a one-dimensional potential energy function and the total energy of a particle, give a qualitative description of the motion of this particle and locate its turning points and regions of acceleration and deceleration.

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1. Introduction

If the potential energy corresponding to a conservative force and the total energy of a particle are known, then all dynamical quantities of the particle can be found. These include the force acting on the particle, the acceleration, the velocity, and the kinetic energy, which along with the total and potential energy, can all be determined at each point in space. The use of potential energy functions and graphs for finding these quantities is the subject of this module. Here we will restrict ourselves to problems that involve only one coordinate, either one-dimensional problems or three-dimensional ones that have spherical symmetry, for which the single coordinate r, the distance of the particle from the force center, is sufficient. The main difference between the two cases is that the coordinate x goes from $-\infty$ to $+\infty$ while the coordinate r goes from 0 to ∞.

2. Force and Potential Energy

2a. E_p from $F(x)$. Given a conservative force acting on a particle, that particle's potential energy function can be obtained from the line integral of the force function along the path taken:[1]

$$E_p(x_2) - E_p(x_1) = -\int_{x_1}^{x_2} F(x)\,dx\,, \tag{1}$$

where $F(x)$ is the force acting on the particle. Then $E_p(x_2)$ is the particle's potential energy function at any point x_2. As usual with potential energies, only differences have any meaning: one normally interprets Eq. (1) as giving a particle's potential energy at any point x_2 relative to its potential energy at some reference point x_1.

[1]See "Potential Energy, Conservative Forces, the Law of Conservation of Energy" (MISN-0-21).

On the other hand, the same integral is often written:

$$E_p(x) - E_p(x_{\text{std ref pt}}) = -\int_{x_{\text{std ref pt}}}^{x} F(x')\,dx'\,,$$

where x' is the "dummy variable of integration." *Help: [S-6]* In fact, this integral is usually written with the standard reference point omitted from the left hand side:

$$E_p(x) = -\int_{x_{\text{std ref pt}}}^{x} F(x')\,dx'\,.$$

The standard reference point used to generate any particular $E_p(x)$ can be deduced by simply seeing what value of x makes E_p zero.

For example, suppose $F(x) = -kx$ so Eq. (1) produces: *Help: [S-7]*

$$E_p(x_2) - E_p(x_1) = k(x_2^2 - x_1^2)/2\,.$$

To simplify this equation we normally choose $x_1 = 0$ and choose E_p to be zero there. Thus we have chosen $x = 0$ as the reference point. Now we replace the position symbol x_2 by the equally good (and simpler) position symbol x. Then we quote: $E_p(x) = kx^2/2$. Any other scientist can see by inspection that we have taken E_p to be zero at $x = 0$.

2b. $F(x)$ from $E_p(x)$. The force acting on a particle can be found from the particle's potential energy function by applying the Intermediate Value Theorem of calculus to Eq. (1).[2] This produces the result:

$$F(x) = -\frac{dE_p(x)}{dx}\,. \tag{2}$$

Thus the force is the negative derivative of the potential energy function. Graphically, Eq. (2) says that at any given point the force is in the direction in which the potential energy decreases and that its magnitude is equal to the slope of the potential energy curve at that point.[3] For example, we can apply Eq. (2) to find the gravitational force near the surface of the earth. Taking the z-axis perpendicular to the surface of the earth and increasing upward, the potential energy of a particle of mass m is:[4]

$$E_p(z) = mgz + \text{ constant.} \tag{3}$$

[2]See Appendix A if interested.
[3]For three-dimensional cases, the force points in the direction in which the potential energy decreases most rapidly and the magnitude of the force is the slope in the direction of most rapid decrease.
[4]See "Potential Energy, Conservative Forces, the Law of Conservation of Energy" (MISN-0-21).

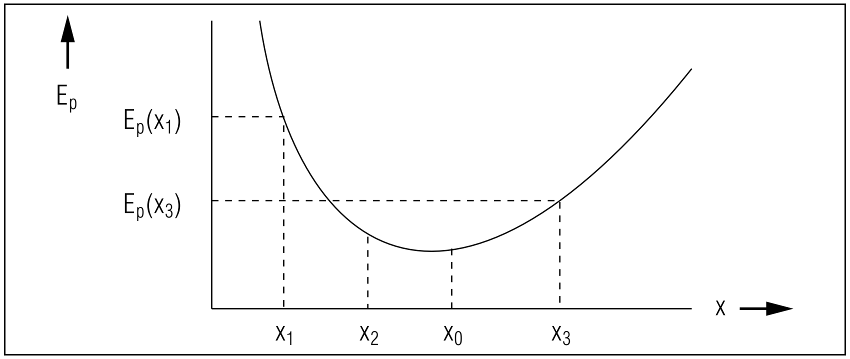

Figure 1. An illustrative potential energy curve (see text).

The force on the particle is then:

$$F(z) = -\frac{dE_p}{dz} = -mg\,. \tag{4}$$

The final minus sign in Eq. (4), with m and g positive, indicates that the force is in the direction in which z decreases, downward.

2c. $F(x)$ from Graph of $E_p(x)$. Characteristics of the force on a particle can be deduced from a graph of the particle's potential energy plotted as a function of position. Such a graph is called a potential energy curve. Consider the potential energy curve in Fig. 1. One immediately notes that the particle with this potential energy is not moving freely. It feels a force which at each point is the negative of the slope of the $E_p(x)$ curve. At all points for which $x < x_0$ the slope of the curve is negative and the particle feels a positive force, one directed to the right. Furthermore, because the slope at x_1 is steeper than at x_2, the magnitude of the force on the particle is greater at x_1 than at x_2. At x_0 the particle feels no force. At all points for which $x > x_0$ the particle feels a negative force, one directed to the left.

3. Deductions From Functional $E_p(x)$

3a. $a(x)$ from $E_p(x)$. Knowing a particle's potential energy as a function of position, one can deduce the particle's acceleration as a function of position. Using Newton's second law, in the form $a(x) = F(x)/m$, and Eq. (2), one gets:

$$a(x) = -\frac{1}{m}\frac{dE_p}{dx}\,. \tag{5}$$

One need only remember Newton's law and Eq. (2) and combine them any time one wants $a(x)$.

3b. $v(x)$ from Conservation of Energy. Although $v(x)$ can be obtained from $E_p(x)$ by integrating $a(x)$,[5] it is generally quicker and easier to use the equation for conservation of energy,

$$\frac{1}{2}mv_0^2 + E_p(x_0) = \frac{1}{2}mv^2(x) + E_p(x)\,. \tag{6}$$

All we need do is rearrange that equation to get:

$$v(x)^2 = v_0(x)^2 - \frac{2}{m}\left[E_p(x) - E_p(x_0)\right]\,. \tag{7}$$

4. Deductions From Graphical $E_p(x)$

4a. Energy Diagrams. If both total energy and potential energy are plotted as functions of position on the same graph, useful information about the motion of the particle may be determined from that diagram.[6] The total energy, potential plus kinetic, is always a horizontal straight line on such a graph because its value is independent of position.[7] Figure 2 shows the potential energy curve of Fig. 1 with two different total energies superimposed on it. For a given total energy the particle's kinetic energy at each point x can be measured as the vertical distance between the potential energy at that point and the total energy. For example, with total energy E_1, the particle's kinetic energy, when it is at position x_A, is the indicated vertical distance shown on the graph. For a particle with total energy E_2, its potential energy at x_A would be the same as in the E_1 case but its kinetic energy would be less.

4b. Changes in Speed. Once the kinetic energy E_k is obtained from the energy diagram, the speed can be found:

$$v = (2\,E_k/m)^{1/2}\,.$$

However, the direction of the particle's velocity is ambiguous: the kinetic energy will match the given E_k whether the particle is moving to the

[5] See Appendix B if interested.

[6] How the various quantities of motion depend upon time cannot be determined from such diagrams.

[7] Equation (6) above, for example, illustrates that for a particle moving in a conservative force field the kinetic plus potential energy of the particle at any one point equals the sum at any other point. Hence the total energy has a constant value; it is conserved.

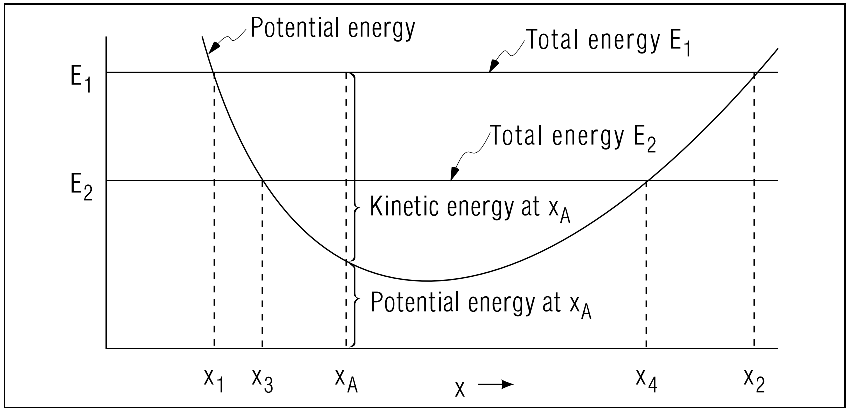

Figure 2. An illustrative energy diagram showing one potential energy and two total energy curves on the same graph. Kinetic energy is the difference between the total and potential energies.

right or to the left. However, the particle's change in kinetic energy, and hence its change in speed, does depend on its direction of motion. For example, if the particle in Fig. 2 is at x_A and is moving to the right, its kinetic energy is increasing so its speed is increasing. Such changes in speed, which depend on the particle's direction of motion, are given by the direction of its acceleration, the negative of the slope of its potential energy curve [Eq. (5)].

4c. Turning Points. There are often limits on the positions available to a particle: those limits are called "turning points" and they depend on the particle's potential energy function and on its total energy. For example: for a particle with the potential energy function shown in Fig. 2, and with total energy E_1, the particle can go no farther to the left than x_1 and no farther to the right than x_2. Thus x_1 and x_2 are the turning points for E_1.

Consider the situation in Fig. 2 with the particle at x_A, with total energy E_1, and moving left. As it moves left its kinetic energy decreases until the particle reaches x_1 where its potential energy equals its total energy. Since nothing is left for its kinetic energy, its velocity at this point is zero. However, the particle is only instantaneously at rest.

Throughout the neighborhood of x_1 the force acts on the particle to the right, as can be seen from the negative slope of E_p in the neighborhood

of x_1. As the particle approaches x_1 from the right, going left, this force causes it to slow down, stop as it reaches x_1, then pick up speed to the right. Thus point x_1 is truly a "turning point" of the motion. Similarly, x_2 is a turning point on the right end of the motion: it is the point at which the particle moving to the right comes to a stop, then starts back left. For total energy E_2 the turning points are x_3 and x_4. Now suppose the total energy line is lowered until it goes through the minimum of the potential energy curve at point x_0: describe the particle's motion for that case. *Help: [S-4]*

5. Dealing with $E_p(r)$

All of the results discussed in this module can be applied to three-dimensional cases that have spherical symmetry. This means cases where the force, potential energy, velocity, etc. depend only on the particle's radius from the origin of coordinates (which is normally the origin of the force associated with E_p). For such cases the one-dimensional position "x" is replaced by the radial position "r." That means that $E_p(r)$ is differentiated with respect to "r" to get $F(r)$, etc. The main difference from Cartesian coordinates is that negative values of r do not exist.

Acknowledgments

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Glossary

- **energy diagram**: diagram consisting of the potential energy curve and a horizontal line representing the total energy of the particle.
- **potential energy curve, potential energy graph**: the graph of the potential energy of a particle as a function of position.
- **turning points**: positions at which a particle changes its direction of motion.

A. Force From Potential Energy

Here we start with the definition of potential energy difference, Eq. (1),

$$E_p(x_2) - E_p(x_1) = -\int_{x_1}^{x_2} F(x)\,dx\,,$$

and invert that equation to derive Eq. (2):

$$F(x) = -\frac{dE_p(x)}{dx}\,.$$

Since the first equation holds for all values of x_1 and x_2 in a given range, we let $x_2 = x_1 + \Delta x$ and write it as:

$$E_p(x_1 + \Delta x) - E_p(x_1) = -\int_{x_1}^{x_1+\Delta x} F(x)\,dx\,.$$

If $F(x)$ is a continuous function, then by the Intermediate Value Theorem[8] there exists an x_0 in the interval $(x_1, x_1{+}\Delta x)$ such that $F(x_0)$ is an average of $F(x)$ over that interval:

$$\frac{1}{\Delta x}\int_{x_1}^{x_1+\Delta x} F(x)dx = F(x_0)\,.$$

Combining the above two equations we get:

$$-\frac{E_p(x_1 + \Delta x) - E_p(x_1)}{\Delta x} = F(x_0)\,.$$

Now we take the limit of both sides as $\Delta x \to 0$. Then, as $\Delta x \to 0$, we see that $x_0 \to x_1$ (*Help: [S-1]*) and $F(x_0) \to F(x_1)$ (*Help: [S-2]*). Then:

$$\lim_{\Delta x \to 0} F(x) = \lim_{\Delta x \to 0}\left[-\frac{E_p(x_1 + \Delta x) - E_p(x_1)}{\Delta x}\right]\,,$$

so:

$$F(x_1) = -\frac{dE_p(x_1)}{dx_1}\,.$$

The equation just above holds for any point x_1 in the given range, so the subscripts "1" can be removed and the proof is finished.

[8]See any calculus textbook; for example, G. Thomas, *Calculus and Analytic Geometry*, third edition, Addison-Wesley (1960), Sections 4-8, 4-9.

B. $v(x)$ and $x(t)$ from $a(x)$

This appendix deals with how to obtain $v(x)$ and $x(t)$ from $a(x)$. Notice the difference from the more straightforward case where all three of those quantities are functions of time. In that case we obtain $v(t)$ and $x(t)$ from $a(t)$ by integrating with respect to time:

$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} a(t)\,dt\,,$$

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)\,dt\,.$$

To obtain $v(x)$ and $x(t)$ from $a(x)$, we first relate v to $a(x)$ through the chain rule[9] to obtain:

$$a(x) = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v\,.$$

Integration of that equation produces the velocity at any point x_2, providing one also knows the particle's speed at some one reference point x_1 (*Help: [S-3]*):

$$\int_{x_1}^{x_2} a(x)\,dx = \frac{1}{2}\left[v(x_2)^2 - v(x_1)^2\right]\,.$$

Although we will not be concerned with it in this module, another integration can be made using $v = dx/dt$ in the form

$$dt = \frac{dx}{v(x)}\,,$$

producing t as a function of x, $t(x)$, and of course a constant of integration will be introduced. The resulting expression can be inverted to give $x(t)$.

[9]See "Review of Mathematical Skills-Calculus: Differentiation and Integration" (MISN-0-1).

PROBLEM SUPPLEMENT

All four problems also occur in this module's *Model Exam.*

1. Consider the potential energy function $E_p(x) = 7x^2 - x^3$. Assume the particle interacting with this potential has mass $M = 60$. For this problem let us neglect units. Draw a rough sketch of $E_p(x)$. *Answer: 18* At a certain instant, the particle is at $x = +1$ with total energy $E = 36$ and is moving to the right.

   a. What is the kinetic energy of the particle at this point? *Answer: 11*

   b. What is its velocity? *Answer: 14*

   c. What is the instantaneous force this particle feels at this point? *Answer: 5*

   d. What is the acceleration (magnitude, direction) this particle undergoes? *Answer: 9*

   e. Therefore, is this particle's speed increasing or decreasing at this point? *Answer: 16* Explain. *Answer: 19*

   f. With this particle moving to the right, and from the shape of the E_p curve in the vicinity of this point ($x = 1$) determine whether the kinetic energy of the particle will increase or decrease. *Answer: 3* Is this consistent with your answer of (e)? *Answer: 12*

   g. For what value of x will the speed of this particle (moving to the right) be zero? (You may not actually be able to find the numerical value of x for which this occurs; however, if you do it right you'll have the equation satisfied by x at this point on the graph you've drawn.) *Answer: 1*

   h. At this point where $v = 0$ what is the direction of the acceleration of this particle? (Get this from determining the sign of the force on the particle as determined from the slope of the curve.) *Answer: 10* With this direction for the acceleration what is the direction of v an instant after the instant when $v = 0$, hence what is the direction of the subsequent motion of the particle? *Answer: 15*

   i. On your $E_p(x)$ plot, show the region of those values of x forbidden to the particle when it has total energy $E = 36$. *Answer: 7*

   j. What are the turning points of the motion of the particle when its total energy is 36? *Answer: 25*

2. Consider the same situation as in Problem 1 except that at point $x = -3.0$ the particle of mass $M = 60$ starts out with speed $v = 0$.

   a. What is its kinetic energy? *Answer: 4*

   b. What is its potential energy? *Answer: 20*

   c. What is its total energy? *Answer: 24*

   d. What is the force (magnitude and direction) on the particle? *Answer: 13*

   e. What is the magnitude of its acceleration? *Answer: 22*

   f. What is the direction of its acceleration? *Answer: 2*

   g. When it gets to $x = 0$, what is its speed? *Answer: 8*

   h. When it gets to $x = 14/3$, what is its speed? *Answer: 21*

   i. What is its acceleration (magnitude and direction) at $x = 14/3$? *Answer: 17*

   j. How far will it go before it turns around? *Answer: 23*

   k. Explain your answer to (j). *Answer: 6*

3. A potential energy function is given by $E_p(r) = -1/2\,kr^2$ where r is the distance from the origin of the coordinate system to a point in space (k is a constant).

   a. Find the magnitude of the force on a particle placed at a point which is at a distance r from the origin. *Answer: 26*

   b. Find the direction of this force. *Answer: 27*

4. a. If the potential energy of a particle is $E_p = k/r$, k being a constant equal to $10\,\mathrm{N\,m^2}$, find the force on the particle when it is at the point $x = 0\,\mathrm{m}$, $y = 1\,\mathrm{m}$, $z = 2\,\mathrm{m}$. *Answer: 28*

   b. What is the direction of the force? *Answer: 29*

Brief Answers:

1. One of the solutions of $x^3 - 7x^2 + 36 = 0$. There are 3 solutions, call them x_1, x_2, x_3 with $x_1 < x_2 < x_3$ (refer to graph, notice x_1 is negative). x_2 is the answer to this question. The actual values are $x_1 = -2$, $x_2 = 3$, $x_3 = 6$.
2. To right.
3. Decreases.
4. Zero.
5. $F_x = -11$ (to left).
6. After point $x = 14/3$, the potential energy continues to decrease to infinity, total energy remains constant, kinetic energy increases, never goes to zero, so v never changes direction.
7. Forbidden regions: $x < x_1$, $x_2 < x < x_3$. All other regions are allowed. The quoted regions are forbidden because the particle would have a negative kinetic energy, hence an imaginary speed in these regions. Verify this from the energy diagram you've sketched.
8. $\sqrt{3}$.
9. $a_x = -11/60$ (to left).
10. To left.
11. $E_k = 30$.
12. Yes, kinetic energy and speed both are decreasing.
13. 69 to right.
14. $v_x = +1$.
15. Because a_x is to left, an instant after $v_x = 0$, v_x is to left.
16. Decreasing.
17. Zero.

18. A rough sketch of $E_p(x)$ can be drawn by considering the following. For very large positive or negative values of x the x^2 term may be neglected compared to the x^3 term. Hence, for very large negative values the graph tends to plus infinity while for very large positive values the graph goes to negative infinity. The graph crosses the x-axis whenever $E_p(x)$ is zero. There are three roots to a cubic equation. In this case the roots are $x = 0$ (twice) and $x = 7$. You can also determine that the graph has a minimum at $x = 0$ and a maximum at $x = 14/3$. With this information a rough graph can be sketched.
19. v_x is positive and a_x is negative so v_x decreases.
20. 90.
21. $1.14 = (529/405)^{1/2}$.
22. 1.15.
23. To infinity.
24. 90.
25. x_1 and x_2: using results of Answer 1 the particle moves between points $x = -2$ and $x = +3$.
26. $+kr$;
27. Outward radially.
28. 2 newtons;
29. Outward, along the radial line going through (0,1,2).

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-Appendix A)*

Since $x_1 \leq x_0 \leq x_1 + \Delta x$,

$$|x_0 - x_1| \leq \Delta x.$$

Then given any $\epsilon > \Delta x$,

$$|x_0 - x_1| < \epsilon.$$

S-2 *(from TX-Appendix A)*

We assumed that $F(x)$ is continuous, so use the definition of continuity.

S-3 *(from TX-Appendix B)*

$$a(x)\,dx = v\,dv$$

$$\int_{x_0}^{x} a(x')\,dx' = \int_{v_0}^{v} v'\,dv' \qquad \textit{Help: [S-5]}$$

$$= \left.\frac{v'^2}{2}\right|_{v_0}^{v}$$

$$= \frac{1}{2}(v^2 - v_0^2)$$

S-4 *(from TX-4c)*

At x_0 the total energy is all in the form of potential energy, hence the velocity of the particle at that point is zero. The acceleration of the particle there is also zero because:

$$F(x_0) = -\left.\frac{dE_p}{dx}\right|_{x_0} = 0.$$

The left and right turning points coincide: the particle cannot move!

S-5 *(from [S-3])*

We changed to primed symbols for the variables of integration because we wanted to use unprimed symbols for the upper limits of integration.

S-6 *(from TX-2a)*

The term “dummy variable of integration” is standard usage. It indicates that the symbol used for the variable of integration is immaterial in a definite integral because it does not appear in the final answer. Thus

$$\int_0^1 x\,dx \qquad \text{and} \qquad \int_0^1 y\,dy$$

give exactly the same answer.

S-7 *(from TX-2a)*

If you are confused by the force being a function of x and the integrand being a function of x', read [S-6]. Merely write the force as a function of the variable required in the integrand, $F(x') = -kx'$, and the integration is easy.

MODEL EXAM

1. See Output Skill K1 in this module's *ID Sheet*.

2. Assume a particle has mass $M = 60$ and potential energy function $E_p(x) = 7x^2 - x^3$. For this problem let us neglect units. Draw a rough sketch of $E_p(x)$. At a certain instant, the particle is at $x = +1$ with total energy $E = 36$ and is moving to the right.

   a. What is the kinetic energy of the particle at this point?

   b. What is its velocity?

   c. What is the instantaneous force this particle feels at this point?

   d. What is the acceleration (magnitude, direction) this particle undergoes?

   e. Therefore, is this particle's speed increasing or decreasing at this point? Explain.

   f. With this particle moving to the right, and from the shape of the E_p curve in the vicinity of this point ($x = 1$) determine whether the kinetic energy of the particle will increase or decrease. Is this consistent with your answer to (e)?

   g. For what value of x will the speed of this particle (moving to the right) be zero? (You may not actually be able to find the numerical value of x for which this occurs: however, if you do it right you'll have the equation satisfied by x at this point and you'll be able to identify the point on the graph you've drawn.)

   h. At this point where $v = 0$ what is the direction of the acceleration of this particle? (Get this from determining the sign of the force on the particle as determined from the slope of the curve.) With this direction for the acceleration what is the direction of v an instant after the instant when $v = 0$, hence what is the direction of the subsequent motion of the particle?

   i. On your $E_p(x)$ plot show the region of those values of x forbidden to the particle when it has this total energy, $E = 36$.

   j. What are the turning points of the motion of the particle when its total energy is 36?

3. Consider the same situation as in Problem 1 except that at point $x = -3.0$ the particle of mass $M = 60$ starts out with speed $v = 0$.

   a. What is its kinetic energy?

   b. What is its potential energy?

   c. What is its total energy?

   d. What is the force (magnitude and direction) on the particle?

   e. What is the magnitude of its acceleration?

   f. What is the direction of its acceleration?

   g. When it gets to $x = 0$, what is its speed?

   h. When it gets to $x = 14/3$, what is its speed?

   i. What is its acceleration (magnitude and direction) at $x = 14/3$?

   j. How far will it go before it turns around?

   k. Explain your answer to (j).

4. A potential energy function is given by $E_p(r) = -1/2\, kr^2$ where r is the distance from the origin of the coordinate system to a point in space (k is a constant).

   a. Find the magnitude of the force on a particle placed at a point which is at a distance r from the origin.

   b. Find the direction of this force.

5. a. If the potential energy is $E_p = k/r$, k being a constant equal to $10\,\mathrm{N\,m}^2$, find the force on the particle when it is at the point $x = 0$, $y = 1$, $z = 2$ (all in meters).

   b. What is the direction of the force?

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 1.

3. See this module's *Problem Supplement*, problem 2.

4. See this module's *Problem Supplement*, problem 3.

5. See this module's *Problem Supplement*, problem 4.